VERSION IV ORIGINAL

ECE 990 Preliminary Report

"Application of Aperture Synthesis Interferometry to Satellite Based 60 GHz Oxygen Line Atmospheric Sounding"

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January 22, 1986

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I. INTRODUCTION

The 60 GHz millimeter wave oxygen absorption line may be utilized to observe the temperature of the atmosphere without the problem of water vapor attenuation from clouds which would otherwise tend to mask areas of the atmosphere from radiometric measurement. To observe wind patterns it is desired to measure the temperature of the oxygen with a resolution of 10 km horizontally as well as vertically.

A satellite based millimeter wave radiometer can obtain this information on more of the atmosphere than a ground based system because of the satellite's unobstructed view of the earth.

A geostationary satellite would provide continuous coverage of a large but restricted area of the earth. Angular resolution would be provided by the antenna system, and radial resolution would be provided by sweeping the frequency of operation across the width of the oxygen absorption line.

Since the satellite is in a fixed position relative to the earth's surface, a continuous communication link with a ground station would permit placement of more of the real time control and data processing on the earth, thus minimizing satellite payload, providing a potential substantial cost savings.

A polar orbiting satellite would permit tomographic techniques to be employed providing coverage of the entire earth, but with only intermittent communications with ground stations, and satellite based data processing requirements would be substantially greater. In this case single frequency operation could be utilized, simplifying hardware requirements since tomography would provide the necessary three dimensional resolution.

Previously, techniques considered to accomplish angular resolution for either of the above orbits involved the use a large mechanical structure antenna, such as a parabolic dish, utilizing gain to obtain resolution.

The image of the atmosphere would then be reconstructed utilizing some form of beam scanning technique. However, aperture synthesis interferometry, long used by radio astronomers, could be utilized instead. This technique has the advantage of providing the same resolution of a large dish antenna, but instead utilizing an array of much smaller antennas having a maximum total spacing equal to the diameter of the dish antenna. Thus a substantial mechanical size and weight advantage may be obtained, resulting in less payload cost.

Aperture synthesis differs from a phased array in that aperture synthesis utilizes individual antenna element spacing rather than total antenna gain to achieve the same angular resolution. Also aperture synthesis observes the entire field of view at once, whereas a phased array (as well as a large parabolic dish antenna) would require some form of beam scanning technique. A phased array achieves resolution in the same way as a parabolic dish antenna, through the use of high gain, and thus still requires the use of a large mechanical structure with a total aperture equivalent to that of the dish.

However, to achieve a substantial payload cost savings over a large mechanical structure antenna, an aperture synthesis array will not have the same gain because of its much smaller total effective antenna capture area. However, signal to receiver noise ratio reduction from reduced gain may be corrected, for a slowly varying object to be imaged radiating thermal noise power, by first utilizing as wide a receiver band width as possible, and then utilizing integration of the cross-correlated signals. This is the technique used by radio astronomers for observation of cosmic thermal sources in the "continuum", or broad bandwidth, mode.

Modern aperture synthesis radiometers require large and complex digital processing. However, by placing most of this computer capability

on the ground, the payload advantage can be maintained with a communications link to a ground station. Much of the real time, "synchronous", computer control could be combined with the "asynchronous" image mapping computer system which would be necessary in any case. Also the error compensation flexibility offered by a computerized system may prove to be of advantage over an analog beam scanned antenna, particularly in the high resolution case.

Electronic phase stability is also a critical requirement of an aperture synthesis instrument. However, techniques can probably be adapted from radio astronomy to meet this requirement. Millimeter wave aperture synthesis radio telescopes are currently in existence.

This preliminary report summaries the results of effort performed from January, 1985 through January, 1986 toward a study of the application of aperture synthesis to this requirement. Chapters II through V describe the technical details and requirements of an aperture synthesis interferometer in general. Chapter VI uses this theoretical background to calculate the approximate antenna requirements for a linear array interferometer in geostationary orbit. Chapter VII discusses a possible system concept in more detail for a spun linear array aperture synthesis interferometer in geostationary orbit. A possible ground-based experiment is also discussed which may also prove useful toward an application involving the detection of wind shear at airport run ways. Near term future goals of my research are also discussed.

The main emphasis of my research is to proceed toward a system design of such a radiometer, and relation of that system design to hardware requirements. My previous hardware design experience at the National Radio Astronomy Observatory's Very Large Array Radio Telescope Project should prove beneficial in this regard.

Most of the work performed thus far has been related to gaining an understanding of aperture synthesis, ambiguities, and array requirements. A preliminary literature search involving more than 60 documents has also been conducted.

II. TWO-ELEMENT INTERFEROMETER WITH COMPLEX CORRELATOR

A microwave interferometer is an instrument that utilizes the interference pattern generated by multiplying the signals from two antennas together to measure angular direction. This section deals with the geometry of such an instrument, along with the calculation of the output or "visibility function" of two classes of interferometer, "simple" and "delay tracking". The delay tracking case provides an advantage over the simple case in that it minimizes the required field of view in the case of the interferometer baseline rotating in the plane of an object to be imaged of relatively small total angular dimension. First, the visibility functions for the simple and delay tracking cases are calculated in terms of delays in Parts A and B. The two visibility functions are then reduced to functions of angle in Part C.

A. SIMPLE INTERFEROMETER "VISIBILITY" FUNCTION IN TERMS OF DELAYS

A simple two-element interferometer consisting of two isotropic, non-directional, antennas spaced a distance S, measured in meters, is shown in Figure II-1. All of the analyses in this paper shall utilize the MKS system of units.

A single received frequency of ω in radians/sec shall be assumed for a point source at infinity. While the simple point source with a single frequency analysis provided here does not provide adequate equations to define resolution, bandwidth effects, or image reconstruction, it can be used to advantage in understanding field of view and antenna array requirements. It is also useful for comparing the field of view performance of a simple interferometer to that of a delay tracking interferometer.

To cover both stationary and rotating situations, the baseline (the line connecting the two antennas) is assumed to be rotating relative to the

SIMPLE
TWO ELEMENT INTERFEROMETER
WITH COMPLEX CORRELATOR

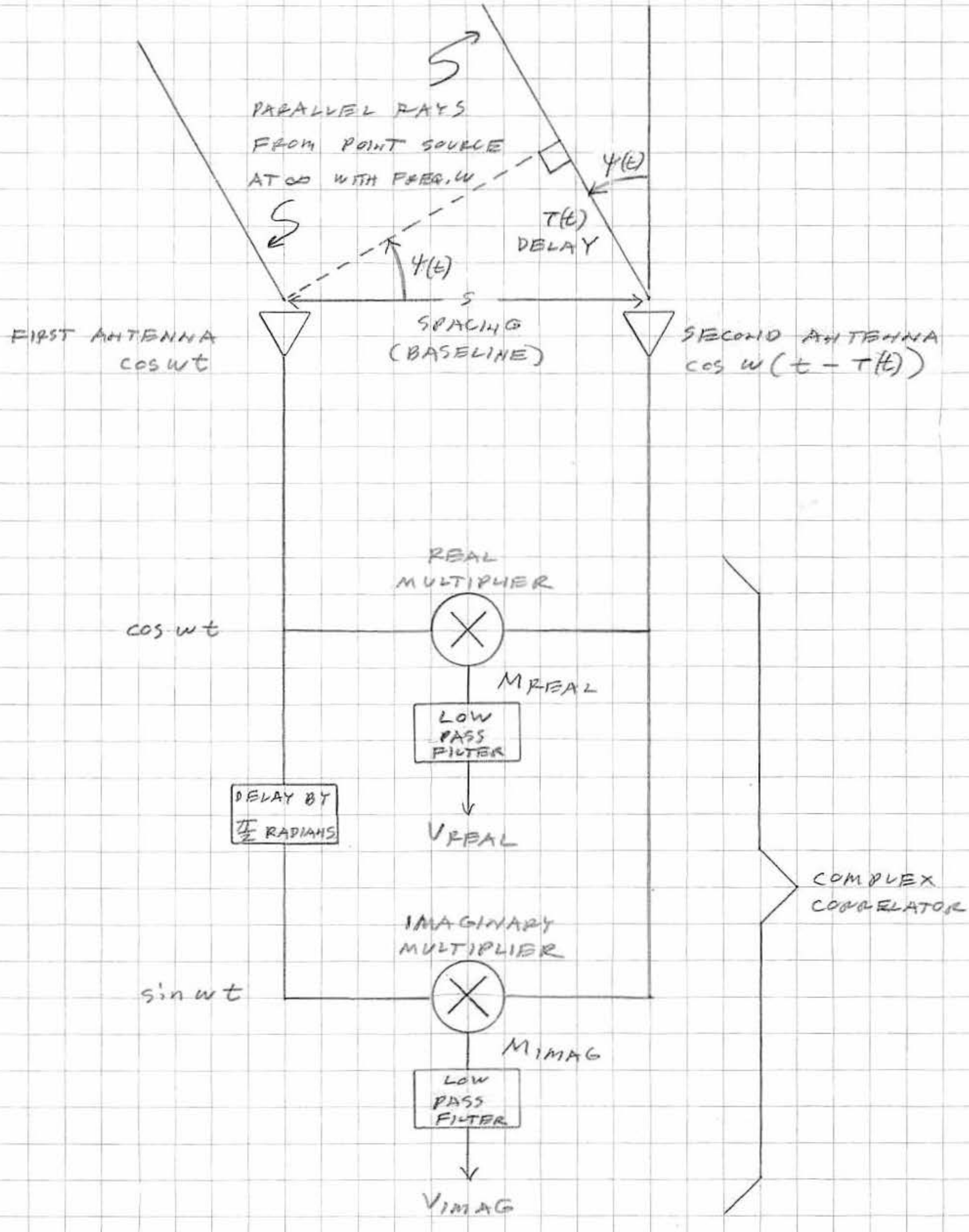


FIGURE II - 1

point source so that the angle to the point source from the normal to the baseline is given by $\psi(t)$ in radians as a function of time.

The delay in arrival at the second antenna relative to the first antenna is given by $\tau(t)$ in seconds as a function of time. Therefore, if the signal present at the first antenna is given by $\cos \omega t$, where ω is the received frequency in radians per second, and t is time in seconds, then the signal at the second antenna will be given by $\cos \omega(t-\tau(t))$. Note that in this simple case, the phase of the received signal at the first antenna is assumed to be 0. For the simple two antenna case shown here, this is satisfactory, since only relative measurements between the two received signals are to be considered.

Assume a complex correlator is employed as shown in Figure II-1. The complex correlator is required to uniquely define the argument of the multiplier outputs, since it provides both the sine and cosine of that argument, both orthogonal functions. The real and imaginary multiplier output calculations shall be handled separately.

Real Multiplier Output

Let M_{real} = the output of the real multiplier.

$$M_{real} = \cos \omega t \cos \omega(t-\tau(t))$$

where t = time in seconds

ω = received frequency in radians/sec

$\tau(t)$ = time delay of arrival of the signal at the second antenna relative to the first antenna in seconds.

From the trigonometric identity, $\cos\alpha \cos\beta = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)$,

$$M_{real} = \tfrac{1}{2}\cos\omega\tau(t) + \tfrac{1}{2}\cos\omega(2t-\tau(t))$$

Drop the high frequency, 2ω, term with a low pass filter to obtain the real part of the complex "visibility" function, V_{real}.

$V_{real} = \frac{1}{2}\cos\omega\tau(t).$

Imaginary Multiplier Output

The imaginary part of the visibility function may be obtained by delaying the phase of one input to the imaginary multiplier by $\pi/2$ radians. Thus, the cos ωt output from the first antenna is delayed in phase by $\pi/2$ radians to provide cos $(\omega t-\pi/2)$. Utilizing the trigonometric identity

$$\cos(\alpha-\pi/2) = \sin\alpha,$$

$$\cos(\omega t-\pi/2) = \sin\omega t$$

Let M_{imag} = the output of the imaginary multiplier.

$$M_{imag} = \cos\omega(t-\tau(t)) \cdot \sin\omega t$$

using the trigonometric identity,

$$\cos\alpha\sin\beta = \tfrac{1}{2}\sin(\alpha+\beta) - \tfrac{1}{2}\sin(\alpha-\beta)$$

$$M_{imag} = \tfrac{1}{2}\sin\omega(2t-\tau(t))$$

$$-\tfrac{1}{2}\sin\omega(-\tau(t))$$

MOVE UP TO SAME LINE

Again, drop the high frequency term with a low pass filter to obtain the imaginary part of the complex "visibility" function, V_{imag}.

$$V_{imag} = -\tfrac{1}{2}\sin\omega(-\tau(t))$$

from the trig identity,

$$\sin(-\alpha) = -\sin\alpha \quad ,$$

$$V_{imag} = \tfrac{1}{2}\sin\omega\tau(t)$$

Therefore, the complex "visibility" function, $V_{complex}$ is expressed in terms of delays as:

$$V_{complex} = V_{real} + jV_{imag}$$

$$V_{complex} = \tfrac{1}{2}\cos\omega\tau(t) + j\tfrac{1}{2}\sin\omega\tau(t)$$

If the $\frac{\pi}{2}$ delay were introduced into the second antenna's signal path instead, it can be shown that the conjugate of the above function will be obtained.

B. DELAY TRACKING INTERFEROMETER "VISIBILITY" FUNCTION IN TERMS OF DELAYS

A delay tracking two element interferometer consisting of two isotropic (nondirectional) antennas spaced a distance s measured in meters is shown in figure II-2. Note the addition of the delay tracking device to the simple interferometer described previously. The significance of this device will be explained in section III.

$\omega \leftarrow$ GREEK LETTER

A single received frequency of ~~w~~ ω radians/sec shall be assumed for a point source at infinity.

To cover both stationary and rotating situations, the baseline (the line connecting the two antennas) is assumed to be rotating relative to the point source so that the angle to the point source from the normal to the baseline is given by $\psi(t)$ in radians as a function of time.

The delay at the second antenna relative to the first antenna is given by $\tau(t)$ in seconds as a function of time. Therefore if the signal present at the first antenna is given by $\cos\omega t$, where ω is the received frequency in radians per second, and t is time in seconds, then the signal at the second antenna will be given by $\cos\omega(t-\tau(t))$.

The first antenna's output, $\cos\omega t$, is delayed in time by $\tau_{ref}(t)$ in seconds by the addition of the delay tracking device. Therefore the signal to the left side of the first multiplier is given by $\cos\omega(t-\tau_{ref}(t))$. Note that $\tau_{ref}(t)$, called the reference delay, is also a function of time because of the relative motion of the baseline. Again assume a complex correlator is employed as shown in figure II-2. The calculation of the real and imaginary multiplier outputs shall again be handled separately.

Real Multiplier Output

Let M_{real} = the output of the real multiplier

$$M_{real} = \cos\omega(t-\tau_{ref}(t)) \cdot \cos\omega(t-\tau(t))$$

DELAY TRACKING
TWO ELEMENT INTERFEROMETER
WITH COMPLEX CORRELATOR

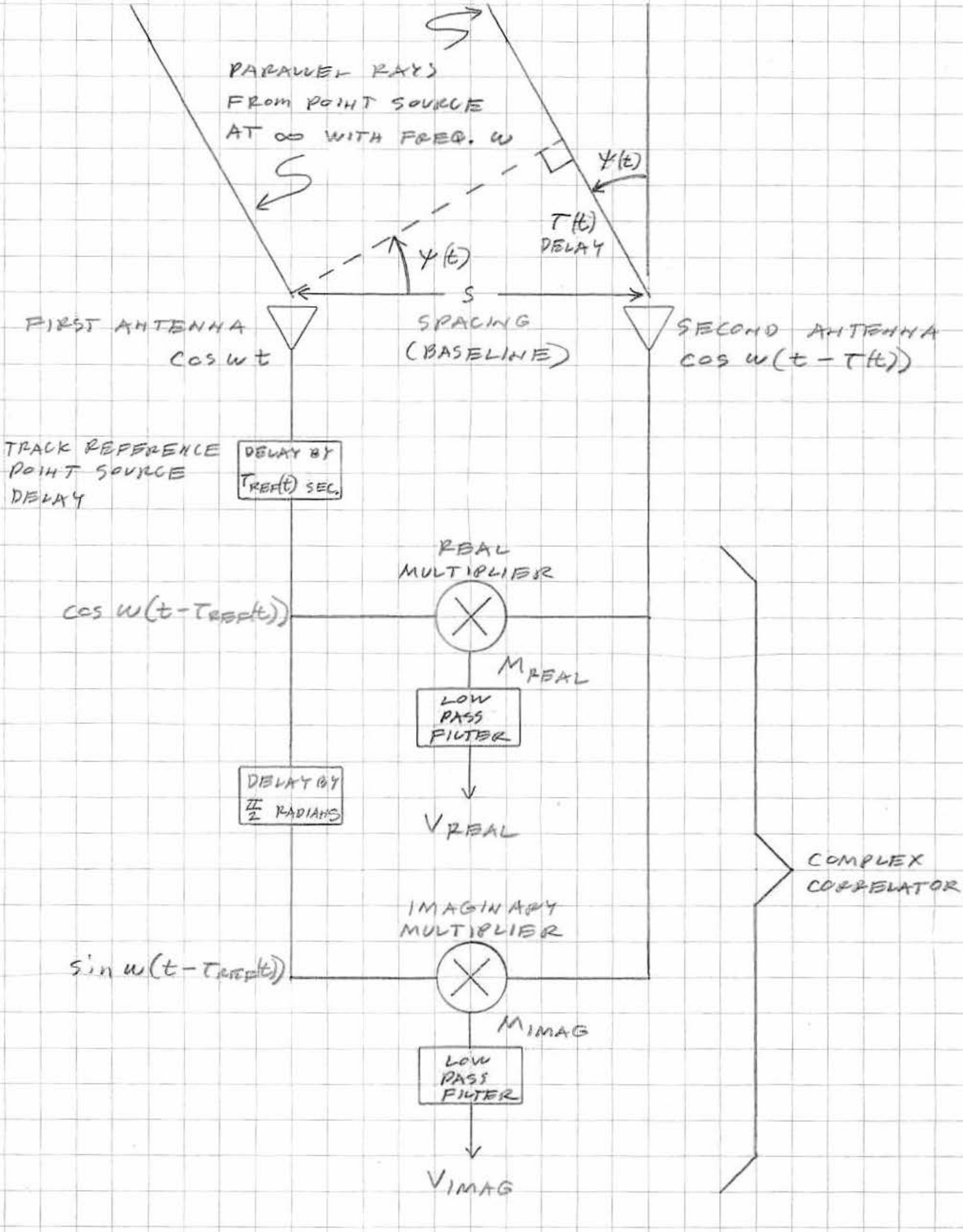


FIGURE II-2

where t = time in seconds

ω = received frequency in radians/seconds

$\tau(t)$ = time delay of arrival at the second antenna relative to the first antenna in seconds

$\tau_{ref}(t)$ = reference time delay in seconds

From the trigonometric identity,

$$\cos\alpha\cos\beta = \tfrac{1}{2}\cos(\alpha-\beta) + \tfrac{1}{2}\cos(\alpha+\beta),$$

$$M_{real} = \tfrac{1}{2}\cos\omega(\tau(t) - \tau_{ref}(t)) + \tfrac{1}{2}\cos\omega(2t - \tau_{ref}(t) - \tau(t))$$

Drop the high frequency, 2ω, term with a low pass filter to obtain the real part of the complex "visibility" function, V_{real}.

$$V_{real} = \tfrac{1}{2}\cos\omega(\tau(t) - \tau_{ref}(t))$$

Imaginary Multiplier Output

The imaginary part of the visibility function may be obtained by delaying the phase of the input to the imaginary multiplier by $\frac{\pi}{2}$ radians. Thus the $\cos\omega(t - \tau_{ref}(t))$ output from the delay tracking network is delayed in phase by $\frac{\pi}{2}$ radians to provide $\cos[\omega(t-\tau_{ref}(t)) - \frac{\pi}{2}]$. Utilizing the trigonometric identity,

$$\cos(\alpha - \tfrac{\pi}{2}) = \sin\alpha,$$

$$\cos[\omega(t - \tau_{ref}(t)) - \tfrac{\pi}{2}] = \sin\omega(t - \tau_{ref}(t))$$

Let M_{imag} = the output of the imaginary multiplier

$$M_{imag} = \cos\omega(t-\tau(t)) \cdot \sin\omega(t-\tau_{ref}(t))$$

Using the trigonometric identity,

$$\cos\alpha\sin\beta = \tfrac{1}{2}\sin(\alpha+\beta) - \tfrac{1}{2}\sin(\alpha-\beta)$$

$$M_{imag} = \tfrac{1}{2}\sin\omega(2t-\tau(t)-\tau_{ref}(t)) - \tfrac{1}{2}\sin\omega(\tau_{ref}(t)-\tau(t))$$

Again, drop the high frequency term with a low pass filter to obtain the imaginary part of the "visibility" function, V_{imag}.

$$V_{imag} = -\tfrac{1}{2}\sin\omega(\tau_{ref}(t) - \tau(t))$$

From the trigonometric identity,

$$\sin(-\alpha) = -\sin\alpha,$$

$$V_{imag} = \tfrac{1}{2}\sin\omega(\tau(t) - \tau_{ref}(t))$$

Therefore the complex "visibility" function, $V_{complex}$ is expressed in terms of delays as:

$$V_{complex} = V_{real} + jV_{imag}$$

$$V_{complex} = \tfrac{1}{2}\cos\omega(\tau(t) - \tau_{ref}(t)) + j\tfrac{1}{2}\sin\omega(\tau(t) - \tau_{ref}(t))$$

Again if the $\frac{\pi}{2}$ delay were introduced into the second antenna's signal path instead, it can be shown that the conjugate of the above function will be obtained.

C. VISIBILITY FUNCTIONS IN TERMS OF ANGLE AND TIME

Referring to figure II-3 for angle relationships to time delays, the visibility function can be expressed in terms of time and angle instead of delays.

Let $\psi(t)$ in radians be the angle measured from the normal to the baseline to any point source as a function of time.

Let $\psi_{ref}(t)$ in radians be the angle from the normal to the baseline to a reference point in the center of the object to be mapped (imaged), again as a function of time. It is not necessary that the reference point be exactly in the center of an object to be mapped, but if it is, a minimum relative field of view will result. "Field of view" is defined as the maximum angular dimension of the source, and is further discussed in chapter III.

Let $\Delta\psi$ in radians be the offset angle from the reference angle $\psi_{ref}(t)$. Note that it is not a function of time.

TWO ELEMENT INTERFEROMETER
ANGLE RELATIONSHIPS

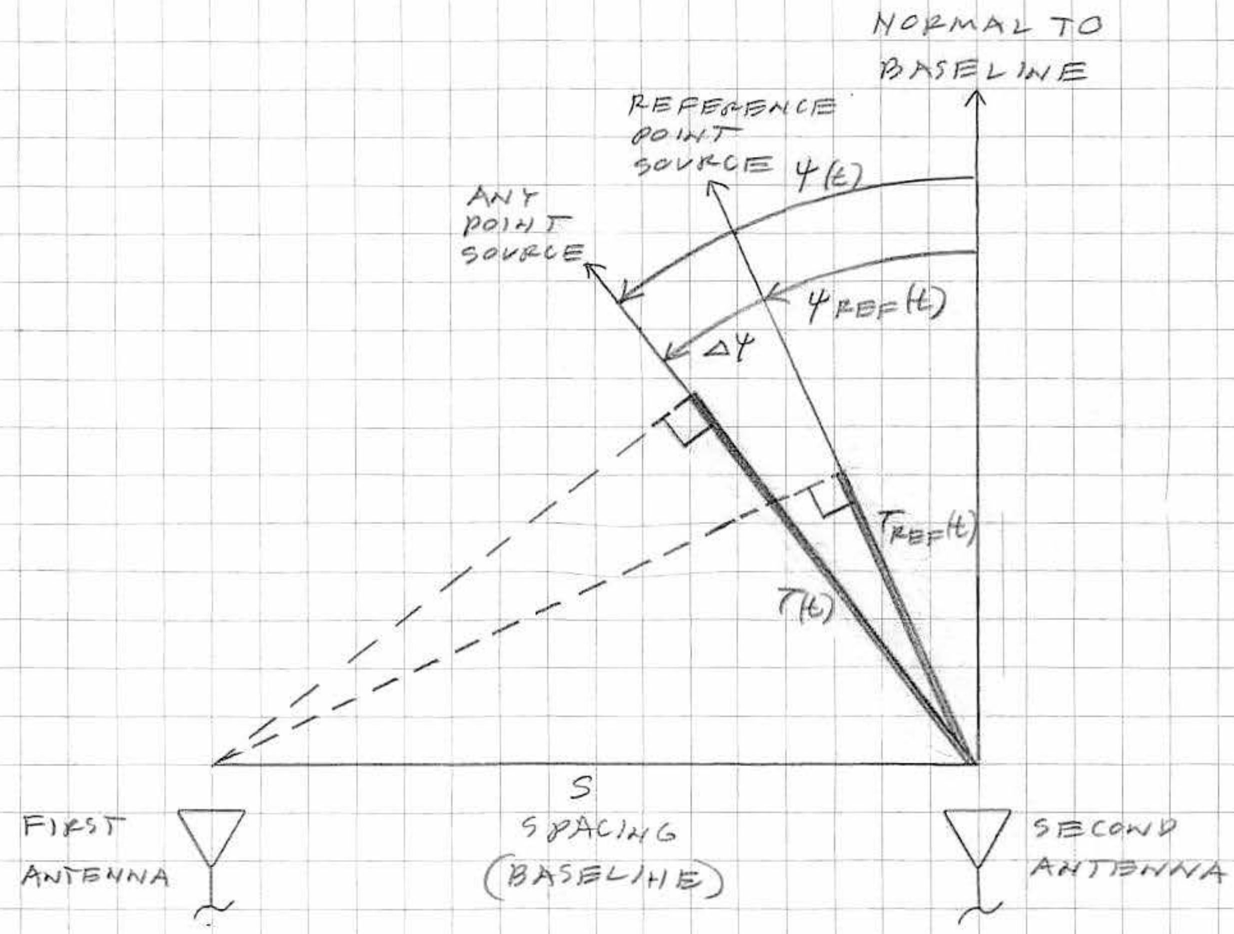


FIGURE II - 3

$$\psi(t) = \psi_{ref}(t) + \Delta\psi$$

where $\psi(t)$ = angle in radians to any point source as a function of time

$\psi_{ref}(t)$ = reference angle in radians as a function of time

$\Delta\psi$ = offset angle in radians of point source relative to reference direction

Let ω_{ref} in radians per second be the angular rate of rotation of the interferometer baseline relative to the reference direction. Therefore,

$$\psi_{ref}(t) = \omega_{ref} \cdot t$$

where $\psi_{ref}(t)$ = reference angle in radians as a function of time

ω_{ref} = angular rate of rotation in radians/second.

t = time in seconds

combining the two equations,

$$\psi(t) = \omega_{ref} \cdot t + \Delta\psi$$

where $\psi(t)$ = angle in radians to any point source

ω_{ref} = angular rate of rotation in radians/second

t = time in seconds

$\Delta\psi$ = offset angle in radians

Let s in meters be the spacing between the two antennas. Let c be the speed of light in meters/sec. Therefore the time delay along the baseline is given by s/c in seconds.

Let $\tau_{ref}(t)$ in seconds be the relative time of arrival delay of the signal at the second antenna relative to the first antenna from an imaginary point source at the reference angle $\psi_{ref}(t)$. From trigonometry, then

$$\tau_{ref}(t) = \frac{s}{c} \sin(\omega_{ref} \cdot t)$$

where $\tau_{ref}(t)$ = relative delay in seconds from an imaginary reference point source

s = spacing in meters

c = speed of light in meters/sec

ω_{ref} = rate of rotation of baseline relative to reference direction in radians/sec.

t = time in seconds

Let $\tau(t)$ in seconds be the time of arrival of the signal at the second antenna relative to the first antenna from any point source.

Similarly then,

$$\tau(t) = \frac{s}{c} \sin(\omega_{ref} \cdot t + \Delta\psi)$$

where $\tau(t)$ = relative delay in seconds from any point source

s = spacing in meters

c = speed of light in meters/sec

ω_{ref} = rate of rotation of baseline relative to reference direction in radians/sec

t = time in seconds

$\Delta\psi$ = offset angle in radians to any point source

The above expressions for delay may now be substituted into the previous visibility equations expressed as functions of delay to obtain the visibility functions in terms of rate of rotation, ω_{ref}, and offset angle, $\Delta\psi$.

Simple Interferometer

$$V_{complex} = \tfrac{1}{2}\cos\omega\tau(t) + j\tfrac{1}{2}\sin\omega\tau(t)$$

substituting for $\tau(t)$,

$$V_{complex} = \tfrac{1}{2}\cos\left[\frac{s\omega}{c} \sin(\omega_{ref} \cdot t+\Delta\psi)\right] + j\tfrac{1}{2}\sin\left[\frac{s\omega}{c} \sin(\omega_{ref} \cdot t+\Delta\psi)\right]$$

Let f in Hertz be the received signal frequency corresponding to ω in radians/second.

$$\omega=2\pi f$$

Substituting for ω,

$$V_{complex} = \tfrac{1}{2}\cos[\frac{2\pi sf}{c}\sin(\omega_{ref}\cdot t+\Delta\psi)] + j\tfrac{1}{2}\sin[\frac{2\pi Sf}{c}\sin(\omega_{ref}\cdot t+\Delta\psi)]$$

For convenience, express the visibility function in terms of the baseline spacing in wavelengths, s_λ, which is dimensionless. Let λ in meters be the wavelength of the received frequency f, in Hertz.

$$s_\lambda = \frac{s}{\lambda}$$

where s_λ = baseline spacing in wavelengths, dimensionless

s = baseline spacing in meters

λ = wavelength of the received frequency, in meters.

But $\lambda = \frac{c}{f}$

$$s_\lambda = \frac{sf}{c}$$

where s_λ = baseline spacing in wavelengths, dimensionless

s = baseline spacing in meters

f = frequency of source in Hertz

c = speed of light in meters/sec.

Substituting back into the previous visibility equation yields:

$$V_{complex} = \tfrac{1}{2}\cos[2\pi s_\lambda \sin(\omega_{ref}\cdot t+\Delta\psi)] + j\tfrac{1}{2}\sin[2\pi s_\lambda \sin(\omega_{ref}\cdot t+\Delta\psi)]$$

where $V_{complex}$ = the complex visibility function for the simple interferometer

s_λ = baseline spacing in wavelengths

ω_{ref} = rate of rotation in radians/sec

t = time in seconds

$\Delta\psi$ = offset angle in radians

Delay Tracking Interferometer

$$V_{complex} = \tfrac{1}{2}\cos\omega(\tau(t) - \tau_{ref}(t)) + j\tfrac{1}{2}\sin\omega(\tau(t) - \tau_{ref}(t))$$

substituting for $\tau(t) = \frac{s}{c}\sin(\omega_{ref}\cdot t+\Delta\psi)$

and $\tau_{ref}(t) = \frac{s}{c}\sin(\omega_{ref}\cdot t)$,

$$V_{complex} = \tfrac{1}{2}\cos(\tfrac{s\omega}{c}[\sin(\omega_{ref}\cdot t+\Delta\psi) - \sin(\omega_{ref}\cdot t)]) + j\tfrac{1}{2}\sin(\tfrac{s\omega}{c}[\sin(\omega_{ref}\cdot t+\Delta\psi) - \sin(\omega_{ref}\cdot t)])$$

Utilizing the trigonometric identity

$$\sin\alpha - \sin\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta),$$

$$V_{complex} = \tfrac{1}{2}\cos[\tfrac{2s\omega}{c}\cos(\omega_{ref}\cdot t+\tfrac{1}{2}\Delta\psi)\sin(\tfrac{1}{2}\Delta\psi)] + j\tfrac{1}{2}\sin[\tfrac{2s\omega}{c}\cos(\omega_{ref}\cdot t+\tfrac{1}{2}\Delta\psi)\sin(\tfrac{1}{2}\Delta\psi)]$$

Again substituting $\omega = 2\pi f$,

$$V_{complex} = \tfrac{1}{2}\cos[\tfrac{4\pi sf}{c}\cos(\omega_{ref}\cdot t+\tfrac{1}{2}\Delta\psi)\sin(\tfrac{1}{2}\Delta\psi)] + j\tfrac{1}{2}\sin[\tfrac{4\pi sf}{c}\cos(\omega_{ref}\cdot t+\tfrac{1}{2}\Delta\psi)\sin(\tfrac{1}{2}\Delta\psi)]$$

Again substituting $s_\lambda = \frac{sf}{c}$,

$$V_{complex} = \tfrac{1}{2}\cos[4\pi s_\lambda\cos(\omega_{ref}\cdot t+\tfrac{1}{2}\Delta\psi)\sin(\tfrac{1}{2}\Delta\psi)] + j\tfrac{1}{2}\sin[4\pi s_\lambda\cos(\omega_{ref}\cdot t+\tfrac{1}{2}\Delta\psi)\sin(\tfrac{1}{2}\Delta\psi)]$$

where $V_{complex}$ = the complex visibility function for the delay tracking interferometer

s_λ = baseline spacing in wavelengths

ω_{ref} = rate of rotation in radians/sec

t = time in seconds

$\Delta\psi$ = offset angle in radians

It can be shown that the delay tracking visibility function reduces to the simple interferometer visibility function for either t=0, the reference direction coincident with the normal to the baseline, or $\omega_{ref}=0$, the stationary case.

Let $\omega_{ref}\cdot t=0$, then

$$V_{complex} = \tfrac{1}{2}\cos[4\pi s_\lambda\cos(\tfrac{1}{2}\Delta\psi)\sin(\tfrac{1}{2}\Delta\psi)] + j\tfrac{1}{2}\sin[4\pi s_\lambda\cos(\tfrac{1}{2}\Delta\psi)\sin(\tfrac{1}{2}\Delta\psi)]$$

Using the trigonometric identity $\cos\alpha\sin\alpha = \frac{1}{2}\sin 2\alpha$,

$$V_{complex} = \tfrac{1}{2}\cos[2\pi s_{\lambda}\sin\Delta\Psi] + j\tfrac{1}{2}\sin[2\pi s_{\lambda}\sin\Delta\Psi]$$

III. SPATIAL APERTURE SYNTHESIS

A. SPATIAL FREQUENCY

The synthesis of a large aperture antenna to achieve high angular resolution imaging in the plane of the baseline of the interferometer can be accomplished by varying the spacing of a two element interferometer and measuring the visibility function as a function of spacing. Spatial frequency is then used with the Nyquist sampling criteria to determine the number of antennas in a "spatial" aperture synthesis interferometer required to meet a given field of view and resolution requirement. The visibility functions for the simple and delay tracking interferometers are given by:

Simple Interferometer

$$V_{complex} = \tfrac{1}{2}\cos[2\pi s_{\lambda}\sin(\omega_{ref}\cdot\tau + \Delta\psi)] + j\tfrac{1}{2}\sin[2\pi s_{\lambda}\sin(\omega_{ref}\cdot\tau + \Delta\psi)]$$

Delay Tracking Interferometer

$$V_{complex} = \tfrac{1}{2}\cos[4\pi s_{\lambda}\cos(\omega_{ref}\cdot\tau + \tfrac{1}{2}\Delta\psi)\sin(\tfrac{1}{2}\Delta\psi] + j\tfrac{1}{2}\sin[4\pi s_{\lambda}\sin(\omega_{ref}\cdot\tau + \tfrac{1}{2}\Delta\psi)\sin(\tfrac{1}{2}\Delta\psi)]$$

where s_{λ} = number of wavelengths baseline spacing, dimensionless

ω_{ref} = rate of rotation in radians/sec

τ = time in seconds

$\Delta\psi$ = offset angle from reference angle in radians

If the spacing in number of wavelengths, s_{λ}, is varied while holding all other variables constant, the real and imaginary components of the visibility functions will vary sinusoidally with s_{λ} with a radian spatial frequency of $\omega_{s\lambda}$. In other words, the visibility functions of both cases of interferometer may be expressed as,

$$V = \tfrac{1}{2}\cos(\omega_{s\lambda}\cdot s_{\lambda}) + j\tfrac{1}{2}\sin(\omega_{s\lambda}\cdot s_{\lambda})$$

Therefore the radian spatial frequency, $\omega_{s\lambda}$, for both cases of interferometer may be expressed as:

<u>Simple Interferometer</u>

$$\omega_{s\lambda} = 2\pi \sin(\omega_{ref} \cdot t + \Delta\psi)$$

<u>Delay Tracking Interferometer</u>

$$\omega_{s\lambda} = 4\pi \cos(\omega_{ref} \cdot t + \tfrac{1}{2}\Delta\psi)\sin(\tfrac{1}{2}\Delta\psi)$$

The spatial frequency, $f_{s\lambda}$ may be defined as

$$f_{s\lambda} = \frac{\omega_{s\lambda}}{2\pi}$$

where $f_{s\lambda}$ is the spatial frequency, dimensionless

$\omega_{s\lambda}$ is the radian spatial frequency in radians

2π has units of radians.

Hence, the visibility functions of both cases of interferometer may be expressed as,

$$V = \tfrac{1}{2}\cos(2\pi f_{s\lambda} \cdot s_{\lambda}) + j\tfrac{1}{2}\sin(2\pi f_{s\lambda} \cdot s_{\lambda})$$

where $f_{s\lambda}$ = spatial frequency, dimensionless

s_{λ} = number of wavelengths spacing, dimensionless

2π = constant with dimension of radians

Therefore the spatial frequency for both cases of interferometer may be expressed as:

<u>Simple Interferometer</u>

$$f_{s\lambda} = \sin(\omega_{ref} \cdot t + \Delta\psi)$$

<u>Delay Tracking Interferometer</u>

$$f_{s\lambda} = 2\cos(\omega_{ref} \cdot t + \tfrac{1}{2}\Delta\psi)\sin(\tfrac{1}{2}\Delta\psi)$$

The spatial frequency, $f_{s\lambda}$, is both a function of time and offset angle $\Delta\psi$, assuming the rotation rate ω_{ref} is a known constant. Since time, t, and rotation rate ω_{ref} are known quantities, the spatial frequency can be measured to determine the position of a source, the offset angle, $\Delta\psi$.

This can be accomplished by either moving one antenna relative to the other to vary s_λ, or by properly spacing many elemental antennas such that the sinusoidal visibility function is sampled adequately. In the latter case, the Nyquist sampling rate which determines the minimum number of antennas required is dependent on the maximum spatial frequency encountered. A comparison of maximum spatial frequencies for both the simple and delay tracking interferometer cases is presented in figure III-1. ψ_{fv} in radians is defined as the total maximum angular extent of the object. The peak angular field of view, $\Delta\psi_{fv}$ in radians is defined as one half the maximum angular extent of the thermal source to be imaged. Thus it represents the full range of $\Delta\psi$ values of sources that the interferometer may observe. For small angular extent sources in a cold sky background the source itself may limit the maximum angular extent encountered. Let ψ_{fv} in radians be defined as the total maximum angular extent of the object to be imaged. In general, ψ_{fv}, should be limited by individual elemental antenna gain, such that other sources (e.g. interference signals from other satellites, cosmic radio sources, the sun, etc.) are prevented from reaching the correlator.

The relationship of $\Delta\psi_{fv}$ to the angular geometry is shown in figure III-2. Both the "restricted" case, for observation of sources of "restricted" field of view and the "general" case, for sources at all possible angles are pictured. Note that the general case is valid for all angles no matter what possible ambiguities may exist. An example of a practical system which exists that validates this approach to the angular extent of the "general" case is that of a narrow aperture direction finder where two interferometers are placed perpendicular to each other to provide cosine and sine components of direction. This type of system is unambiguous, but

MAXIMUM SPATIAL FREQUENCIES

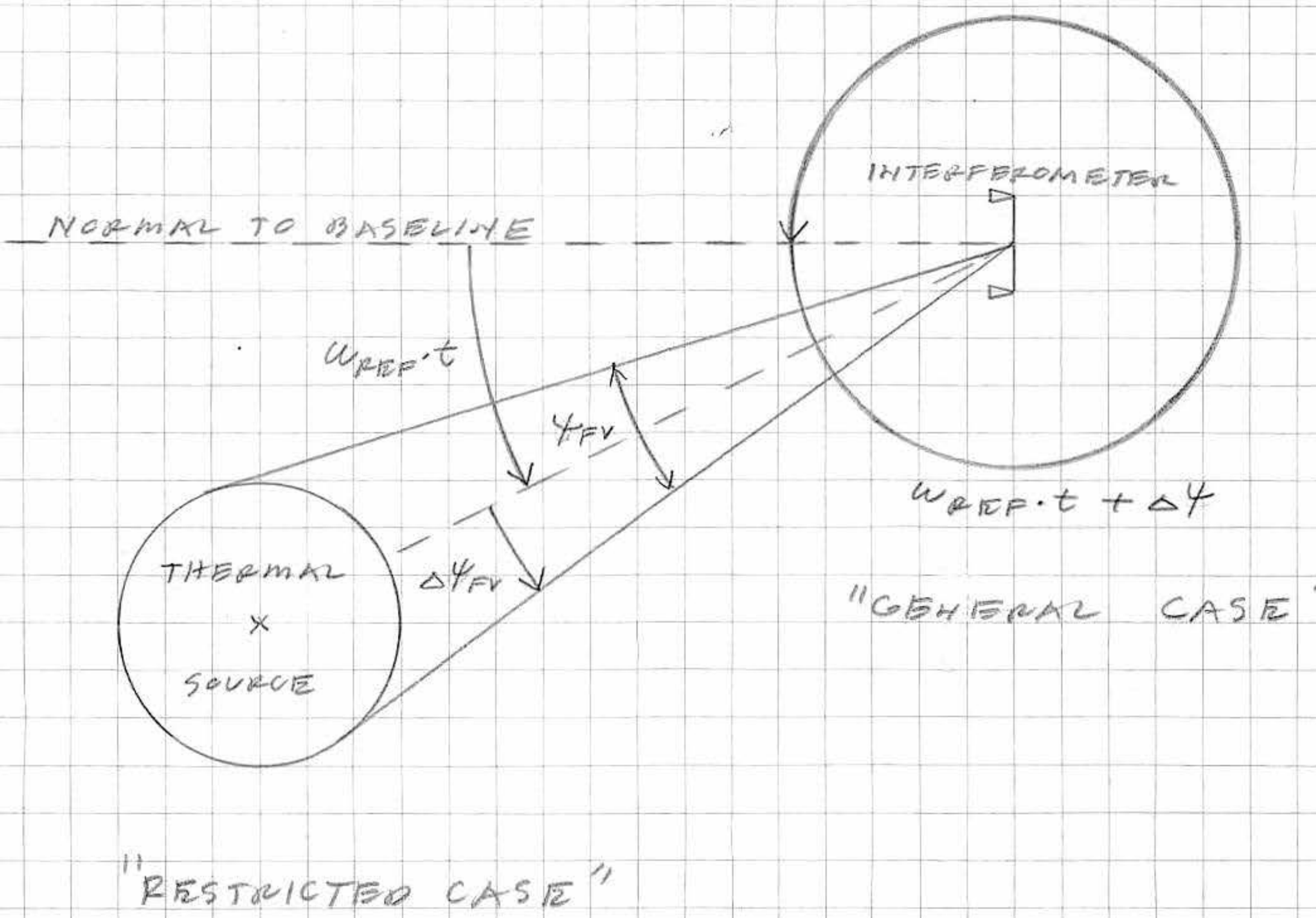
SIMPLE $f_{s\lambda} = \sin(\omega_{ref} \cdot t + \Delta\psi)$

CASE	$\omega_{ref} \cdot t$	$\Delta\psi_{max}$	$f_{s\lambda max}$
General Rotating	Any	Any	1
Restricted Rotating	Any	$\Delta\psi_{fv}$	1
General Stationary	0	Any	1
Restricted Stationary	0	$\Delta\psi_{fv}$	$\sin(\Delta\psi_{fv})$

DELAY TRACKING $f_{s\lambda} = 2\cos(\omega_{ref} \cdot t + \frac{1}{2}\Delta\psi)\sin(\frac{1}{2}\Delta\psi)$

CASE	$\omega_{ref} \cdot t$	$\Delta\psi_{max}$	$f_{s\lambda max}$
General Rotating	Any	Any	2
Restricted Rotating	Any	$\Delta\psi_{fv}$	$2\sin(\frac{1}{2}\Delta\psi_{fv})$
General Stationary	0	Any	1
Restricted Stationary	0	$\Delta\psi_{fv}$	$\sin(\Delta\psi_{fv})$

FIGURE III-1



FIELD OF VIEW, $\Delta\psi_{FV}$, FOR RESTRICTED AND GENERAL CASES

FIGURE III - 2

is capable of observing signals at all angles, even though by symmetry each interferometer is ambiguous. Ambiguities are discussed in more detail in Chapter IV.

The importance of delay tracking is indicated in figure III-1 by a comparison of $f_{s\lambda}$ max between the simple and delay tracking interferometers for the restricted rotating case. For sources of small angular extent in a rotating system, delay tracking is seen to provide an enormous advantage over the simple type of interferometer, reducing the spatial frequency by a factor of $2 \sin(\frac{1}{2}\Delta\psi_{FV})$.

B. RESOLUTION AND ARRAY LENGTH

From the resolution required for a given application, the maximum linear dimension, or aperture, can be calculated for an aperture synthesis interferometer.

Resolution is defined as the ability to make distinguishable the individual parts of an object. Christiansen and ~~B~~Hogbom state that the Rayleigh convention for the resolution of an optical instrument is not suitable for a radio telescope.[1] Johnson and Jasik state that two sources will be resolved if the half power beamwidth of the antenna system is less than one half the beamwidth between the first nulls.[2] This is stated to usually be the case. Therefore the half power beamwidth may be utilized as the limiting resolution case of a radio telescope.[1] Verschuur and Kellerman give the minimum half power beamwidth of a minimum redundancy linear array (as proposed in section IIIC to minimize hardware requirements) as the wavelength divided by the maximum linear dimension of the array.[3] Therefore,

$$\psi_{resmin} = \frac{\lambda}{s_{max}}$$

where ψ_{resmin} = minimum resolution angle in radians

λ = wavelength in meters

s_{max} = maximum linear dimension of the array in meters

In terms of the maximum array spacing in number of wavelengths,

$$s_{\lambda max} = \frac{s_{max}}{\lambda}$$

where $s_{\lambda max}$ = maximum linear dimension of array in numbers of wavelengths, dimensionless

s_{max} = maximum linear dimension of array in meters

λ = wavelength in meters

Therefore, the requirement for the maximum dimension of the array is given by,

$$s_{\lambda max} = \frac{1}{\psi_{resmin}}$$

where $s_{\lambda max}$ = maximum linear dimension of array in numbers of wavelengths, dimensionless

ψ_{resmin} = minimum resolution angle in radians

and the maximum array length or aperture can be calculated using,

$$s_{max} = \frac{\lambda}{\psi_{resmin}}$$

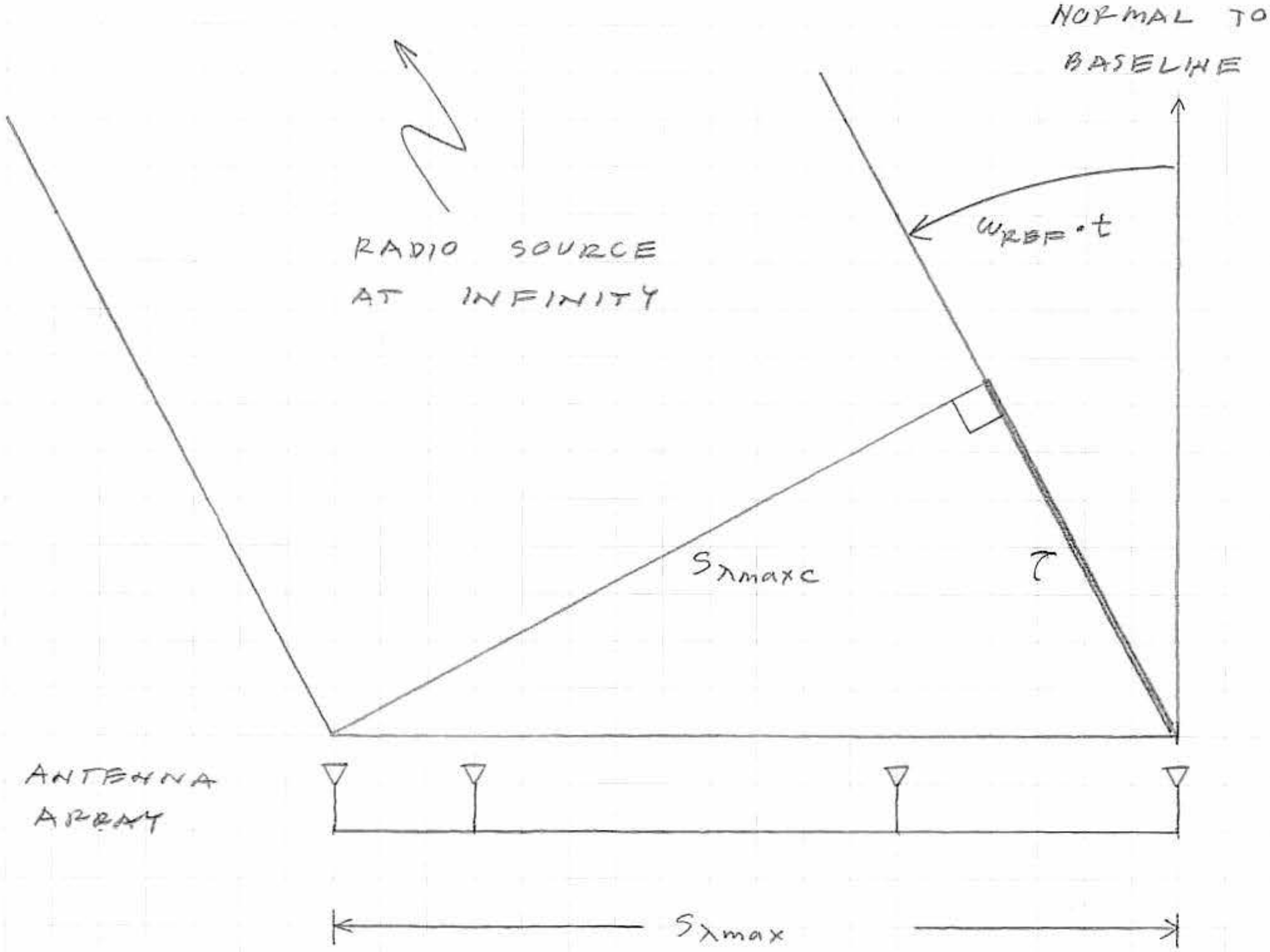
where s_{max} = maximum array length in meters

λ = wavelength in meters

ψ_{resmin} = minimum resolution angle in meters.

Rogers states that angular resolution varies with direction by the component of the maximum baseline normal to the source direction.[4] Figure III-3 shows that the component to the baseline is given by

$$s_{\lambda maxc} = s_{\lambda max}\cos(\omega_{ref} \cdot t)$$



COMPONENT OF MAXIMUM ANTENNA SPACING NORMAL TO THE DIRECTION OF THE SOURCE

FIGURE III - 3

where $s_{\lambda maxc}$ = component of maximum antenna spacing normal to the direction of the source in numbers of wavelengths, dimensionless

$s_{\lambda max}$ = maximum antenna separation in numbers of wavelengths, dimensionless.

ω_{ref} = angular rate of rotation of array in radians/sec.

t = time in seconds.

$$\psi_{res} = \frac{1}{s_{\lambda maxc}}$$

where ψ_{res} = resolution angle in radians in any reference direction.

$s_{\lambda maxc}$ = component of maximum antenna spacing normal to the direction of the source in numbers of wavelengths, dimensionless.

Note that the resolution angle has been selected to be in the reference direction usually to the center of the object to be imaged. For a satellite mapping the Earth's atmosphere this is sometimes called the "well" resolution angle. Therefore,

$$\psi_{res} = \frac{1}{s_{\lambda max} \cdot \cos(\omega_{ref} \cdot t)}$$

where ψ_{res} = resolution angle in radians in any reference direction.

$s_{\lambda max}$ = maximum antenna spacing in numbers of wavelengths, dimensionless.

ω_{ref} = rate of rotation in radians/second of antenna array in plane of source.

t = time in seconds.

Note that the minimum resolution angle occurs at $\omega_{ref} \cdot t = 0$, therefore for the calculation of array requirements it shall be assumed that either the normal to the antenna baseline points toward the source or that at some time this situation still occurs, in order to minimize microwave as well as digital processing hardware requirements.

C. LINEAR ARRAY ANTENNA REQUIREMENTS

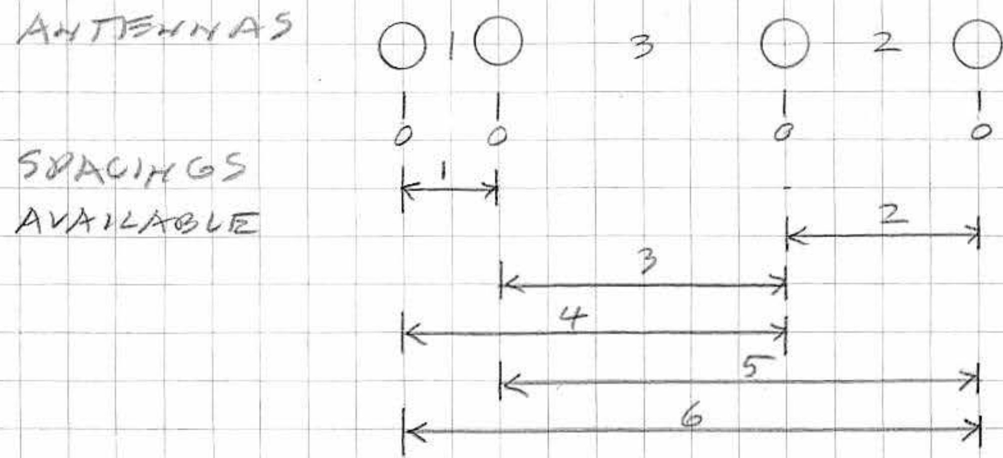
Utilizing information on the field of view and resolution required along with the previous information on the maximum spatial frequency to be encountered one can predict the number of antennas required for a linear array aperture synthesis interferometer.

If a finite number of elemental antennas are used in a linear array, the spatial domain will be sampled as shown in figure III-4 which shows a zero redundancy configuration and spacings for 4 antennas. It is assumed that the output from each antenna is multiplied by the outputs from all of the other antennas to obtain the spacings required. A plot of the real part of the sinusoidal visibility function is superimposed on the diagram to illustrate the Nyquist sampling requirement in relation to the spatial frequency. The Nyquist sampling theorem requires that each visibility function spatial period must be sampled at least two times to insure no loss of information.

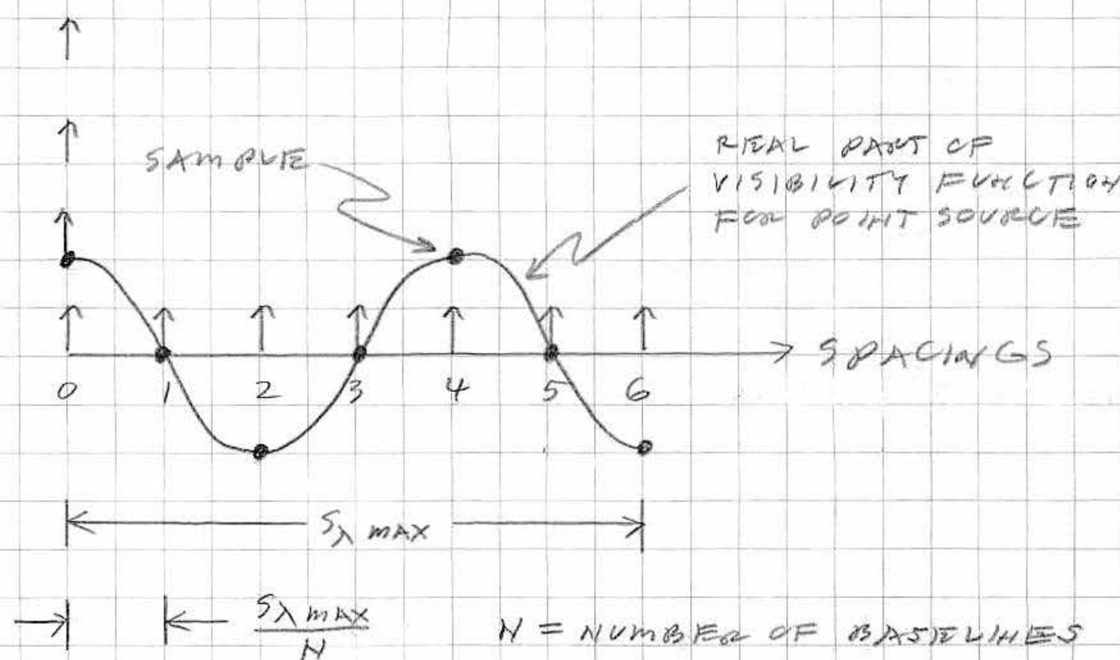
Note that in practice, the required sampling rate will be somewhat higher. Adapting Stremler's time signal analysis to the spatial signal analysis here, two limitations raise the required sampling rate.[5]

First the spatial signal that is being sampled is spatially bounded, from $s_{\lambda min}$ to $s_{\lambda max}$, not from $-\infty$ to $+\infty$, and a spatially limited signal can never be spatially frequency limited. Second a perfect elemental antenna cannot be built that would be capable of totally rejecting spatial frequencies above $f_{s\lambda max}$.

Both of these limitations will thus require a somewhat higher sampling rate to achieve given desired worst case aliasing errors. Therefore the following analysis is in reality a best case analysis for N_{min}, the minimum number of required samples, as well as for A_{min}, the minimum number of required elemental antennas.



SAMPLES AVAILABLE IN THE SPATIAL DOMAIN.



SAMPLING IN THE SPATIAL FREQUENCY DOMAIN FOR A ZERO REDUNDANCY LINEAR ARRAY

FIGURE III - 4

The Nyquist sampling theorem can be expressed albegraically as:

$$2\left(\frac{s_{\lambda max}}{N_{min}}\right) < P_{s\lambda min}$$

where $\frac{s_{\lambda max}}{N_{min}}$ = spatial width of each sample, dimensionless

$s_{\lambda max}$ = maximum antenna spacing, aperture, in numbers of wavelengths, dimensionless

N_{min} = minimum number of non-redundant antenna spacings, dimensionless

$P_{s\lambda min}$ = minimum spatial period of visibility function, dimensionless

Therefore $N_{min} > \frac{2s_{\lambda max}}{P_{s\lambda min}}$

since $f_{s\lambda max} = \frac{1}{P_{s\lambda min}}$,

$$N_{min} > 2\, s_{\lambda max} f_{s\lambda max}$$

where N_{min} = minimum number of non-redundant antenna spacings, dimensionless

$s_{\lambda max}$ = maximum antenna spacing, aperture, in numbers of wavelengths, dimensionless

$f_{s\lambda max}$ = maximum spatial frequency of visibility function, dimensionless

The maximum number of "non-redundant" spacings or minimum number of samples to satisfy the sampling theorem for 1 to 4 antennas is given by A. T. Moffett[6] as

$$N_{min} = \frac{A(A-1)}{2}$$

where N_{min} = maximum number of non-redundant spacings available from a given number of antennas or minimum number of samples required to satisfy the Nyquist sampling theorem, dimensionless.

A = total number of elemental antennas.

Unfortunately, 1 to 4 antennas are the only linear arrays having no redundancy[6]. Above four antennas, the spacings obtained will not be evenly spaced, no matter what algorithm is chosen to space them. However "minimum redundancy" arrays can be chosen. An example of a minimum redundancy array of 5 antennas is shown in Figure III-5. In this case one of the baseline spacings is redundant thus only providing 9 nonredundant spacings, whereas 10 total possible spacings are available from A(A-1)/2.

Therefore, a redundancy number, R, is defined using the equation,

$$N_{min} = \frac{A(A-1)}{2R}$$

where N_{min} = minimum number of non-redundant spacings required to satisfy the Nyquist sampling criteria.

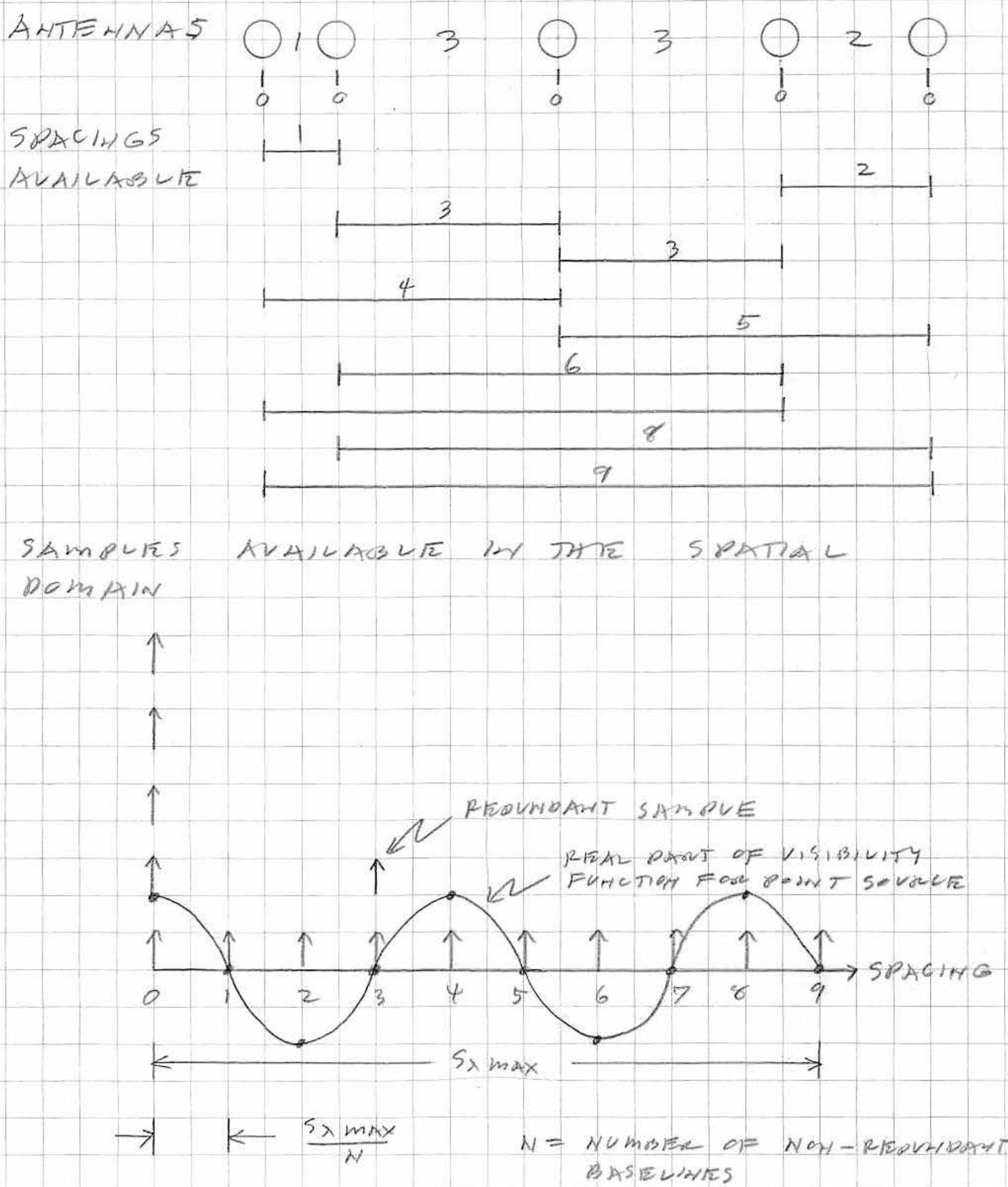
A = number antennas

R = redundancy

In the above example of five antennas, R is calculated to be 10/9 or 1.11.

Figure III-6 tabulates values of R for minimum redundancy.[6] For larger linear arrays of antennas than 11, the redundancy has been proven to be no less than 1.217 and no greater than 1.332[6].

Note that the equality for N_{min} will also be valid for non-integer spacings if one takes N_{min} to be the ratio of the greatest spacing, the aperture, to the greatest sized incremental spacing. The greatest sized incremental spacing is the worst case spacing to meet the given Nyquist sampling requirement for the maximum visibility function spatial frequency requirement based on the angular field of view of the source to be imaged.



MINIMUM REDUNDANCY ARRAY OF 5 ANTENNAS

FIGURE III - 5

PLEASE TYPE

MINIMUM REDUNDANCY ARRAY CONFIGURATIONS

A	N_{min}	R	CONFIGURATION
1	0	1	•
2	1	1	•1•
3	3	1	•1•2•
4	6	1	•1•3•2•
5	9	1,11	•1•3•3•2•
6	13	1.16	•1•5•3•2•2•
7	17	1.24	•1•3•6•2•3•2•
8	23	1,22	•1•3•6•6•2•3•2•
9	29	1.24	•1•3•6•6•6•2•3•2•
10	36	1,25	•1•2•3•7•7•7•4•4•1•
11	43	1,30	•1•2•3•7•7•7•7•4•4•1•
>11		$1,217 \leq R \leq 1,332$	

ADAPTED FROM MOFFETT[6]

FIGURE III – 6

Utilizing some algebra and the quadratic formula, an equation can be found for the number of antennas, A, to produce a required minimum number of non-redundant spacings, N_{min} to satisfy the sampling theorem requirement.

$$N_{min} = \frac{A(A-1)}{2R}$$

$$2RN_{min} = A^2 - A$$

$$A^2 - A - 2RN_{min} = 0$$

Using the quadratic formula to solve the above equation,

$$a(A)^2 + b(A) + c = 0 \quad , \; A = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$a = 1, \; b = -1, \; c = -2RN_{min}$$

$$A = \frac{1 \pm \sqrt{1 - 4(1)(-2RN_{min})}}{2}$$

$$A = \frac{1 \pm \sqrt{1 + 8RN_{min}}}{2}$$

Since only positive and integer values of A are of interest, the minus sign can be dropped, and the integer function can be used to determine A_{min}, the minimum integer number of antennas required to produce at least a required minimum number of spacings. The integer function will round down, therefore 1 must be added to it to insure an adequate number of antennas. Therefore,

$$A_{min} = 1+\mathrm{INT}\left[\frac{1 + \sqrt{1 + 8RN_{min}}}{2}\right]$$

where A_{min} = minimum integer number of antennas (dimensionless) required to produce at least a required minimum number of non-redundant spacings, N_{min}.

INT = integer function

R = redundancy, dimensionless

N_{min} = required minimum number of spacings to satisfy the Nyquist sampling theorem requirement.

For a large number of antennas, there will almost always be a greater number of non-redundant spacings available than are required because of the sum of 1 with the integer function as well as the square root. Previously, it was found that $N_{min} > 2s_{\lambda max} f_{s \lambda max}$. Therefore, it is valid to drop the greater than sign for all practical purposes for large arrays. Therefore let,

$$N_{min} = 2s_{\lambda max} f_{s \lambda max}$$

for substitution into the previous antenna formula. Substituting for N_{min} then yields,

$$A_{min} = 1 + \mathrm{INT}\left[\frac{1 + \sqrt{1 + 16Rs_{\lambda max} f_{s \lambda max}}}{2}\right]$$

where A_{min} = minimum integer number of antennas, dimensionless, required to adequately sample a given spatial frequency

INT = integer function

R = redundancy, dimensionless, see Figure III-6 for values

$s_{\lambda max}$ = maximum dimension of array in number of wavelengths, dimensionless

$f_{s \lambda max}$ = maximum spatial frequency, dimensionless

However, the resolution angle requirement determines $s_{\lambda max}$, the maximum linear dimension or aperture of the array. This relation was previously derived as

$$s_{\lambda max} = \frac{1}{\psi_{resmin}}$$

Substituting into the minimum antenna equation,

$$A_{min} = 1+\text{INT}\left[\frac{1+\sqrt{1+\frac{16Rf_{s\lambda max}}{\psi_{resmin}}}}{2}\right]$$

$f_{s\lambda max}$ is dependent on the field of view requirements as stated in Figure III-1. Thus the dependence of A_{min} on $\Delta\psi_{fv}$ may be derived using whichever one of the eight cases listed in the table of figure III-1 is appropriate. Because of the expense of microwave and digital processing hardware and ambiguity considerations presented in section IV, only three of the eight cases would be considered practical. The first two of these are included in the restricted field of view stationary case, which results in the same $f_{s\lambda max}$ for either the simple or the delay tracking interferometers where,

$$f_{s\lambda max} = \sin\Delta\psi_{fv}$$

The third practical case is the restricted field of view rotating case of the delay tracking interferometer, which would be the minimum spatial frequency case for a rotating interferometer. In this case,

$$f_{s\lambda max} = 2\sin\tfrac{1}{2}\Delta\psi_{fv}$$

Substitution results in the final equations relating the number of antennas required for given worst case peak field of view and resolution angle requirements. Note that these should be considered best case design equations for the reasons stated earlier.

Restricted Stationary

Simple or Delay Tracking Interferometer

$$A_{min} = 1+\text{INT}\left[\frac{1+\sqrt{1+\frac{16R\sin\Delta\psi_{fv}}{\psi_{resmin}}}}{2}\right]$$

Restricted Rotating

Delay Tracking Interferometer

$$A_{min} = 1+\mathrm{INT}\left[\frac{1+\sqrt{1+\dfrac{32R\sin\frac{1}{2}\Delta\psi_{fv}}{\psi_{resmin}}}}{2}\right]$$

where A_{min} = minimum integer number of antennas, dimensionless, for given field of view and resolution requirements

INT = integer function

R = redundancy of linear array, dimensionless, see figure III-6 for values

$\Delta\psi_{fv}$ = worst case peak field of view in radians

ψ_{resmin} = minimum resolution angle in radians.

These two equations predicting the number of antennas required for a linear array, along with the equation relating the maximum linear dimension or aperture of the array,

$$s_{max} = \frac{\lambda}{\psi_{resmin}}$$

where s_{max} = maximum spacing or aperture of array in meters

λ = wavelength in meters

ψ_{resmin} = minimum resolution angle in radians

minimum

define the maximum complexity and size of the array. Possible practical situations are presented in Chapter VI.

D. ELEMENTAL ANTENNA REQUIREMENTS

The beamwidth of each elemental antenna will generally be approximately equal to the field of view, since this will result in the minimum spatial frequency visibility function to be sampled. Elemental antenna

gain will affect the signal to noise ratio received, but in general, this is not significant, if the time required for integration can be tolerated, since integration of the cross-correlated signals may be employed to generally obtain any desired signal to noise ratio for imaging of thermal noise sources.

Thus an aperture synthesis interferometer can achieve a substantial mechanical structure advantage over a parabolic or phased array antenna for the same resolution requirement.

In a practical situation severe constraints may be placed on the elemental antennas in terms of how rapidly they reject interfering signals outside of the field of view. These signals may arise from the sun, other satellites, cosmic radio sources, etc. Since they are outside the field of view they will not be sampled adequately in the spatial ~~frequency~~ domain and may be aliased into the image. The elemental antenna requirements thus have to be determined by an error analysis done on the actual application and type of interferometer employed.

E. FOURIER TRANSFORM PROCESSING

While the previous analyses covered only a single frequency point source, in a practical situation infinitely many such sources are present simultaneously. A discrete Fourier transform is then utilized to sort out the different resolution elements from which an image of the entire object being observed is constructed. This provides the basis for the "spatial" approach to aperture synthesis. Rotation of a two element interferometer with time may also be utilized but this has not been examined as yet.

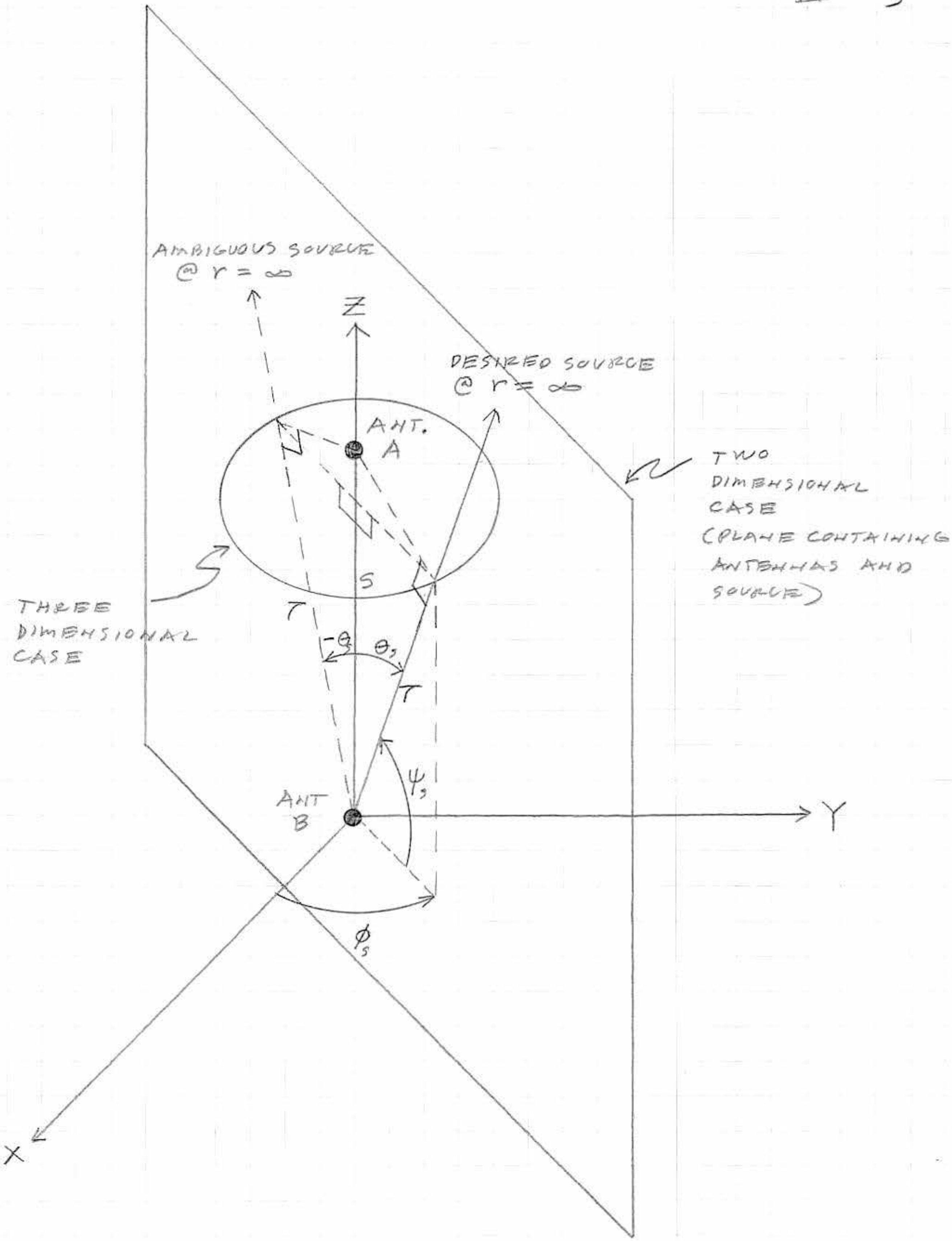
IV. AMBIGUITIES

Ambiguities in the angular position of a source measured with respect to the simple two element interferometer may result from several independent causes. Ambiguities due to geometry, the type of correlator utilized with a given baseline spacing, undersampling for a given field of view, and Fourier transform "windowing" are discussed in this section.

A. GEOMETRY

Ambiguities in the delay function $\tau(t)$ for the total angle $\omega_{ref} \cdot t + \Delta\psi$ exist due to the symmetry of the geometry of the simple two element interferometer with isotropic (non-directional) antennas. Figure IV-1 illustrates the ambiguities due to geometry for both the three dimensional case as well as the more restricted two dimensional case. The analyses for both cases assume that the time of arrival difference between the two antennas can be measured unambiguously by the rest of the interferometer system. This assumption isolates the ambiguities due to geometry from ambiguities due to other causes. Figure IV-1 is drawn consistent with standard engineering notation for a right hand spherical coordinate system. r is defined as the radius to the source in meters, θ is defined as the angle from the positive z-axis to the source, and ϕ is defined as the angle from the x axis to the projection of the source in the x-y plane.

In this analysis, both the desired source and any undesired ambiguous sources are assumed to be at infinity. The coordinate system used in the previous planar analyses is maintained with the use of ψ defined as the angle measured from the perpendicular to the antenna baseline to the source in the plane defined by the two antennas and the source. The delay function τ is defined as the delay in the time of arrival of the signal from the source at antenna B relative to antenna A.



GEOMETRIC AMBIGUITIES IN TWO AND THREE DIMENSIONS

FIGURE IV - 1

1. Three Dimensional Ambiguities

From Figure IV-1, the desired source is located at coordinates ($r=\infty$, $\theta=\theta_s$, $\phi=\phi_s$). The delay function τ is given by:

$$\tau = \frac{s}{c} \cos \theta_s$$

where τ = delay of time of arrival of signal at antenna B relative to antenna A

s = spacing between the two antennas in meters.

c = speed of light in meters/sec.

θ_s = angle to source measured from the positive z axis.

Note that τ is independent of ϕ. Therefore, any signal at coordinates ($r=\infty, \theta=\theta_s, \phi\neq\phi_s$) is ambiguous with the desired source at ($r=\infty, \theta=\theta_s, \phi=\phi_s$) since exactly the same time of arrival delay τ is achieved. Directions of sources resulting in identical delays in three dimensions are indicated by the conical surface containing antenna B and the circle shown in Figure IV-1. Note that these directions are the only ambiguities resulting from geometry. Sources at $r=\infty$ and contained within only the one conical surface given above will result in identical delays. Any source at ($r=\infty, \theta=\pi-\theta_s$, any ϕ) will result in a negative delay and so will not be ambiguous.

The ambiguous sources described by coordinates ($r=\infty, \theta=\theta_s, \phi\neq\phi_s$) result from the cyclindrical symmetry of the simple two-element array. Thus the addition of more elements to the linear array does not aid this problem since array symmetry is maintained. Also directional antennas cannot eliminate three dimensional ambiguities since $\phi\neq\phi_s$ can be arbitrarily close to $\phi=\phi_s$, the direction of the desired source.

Two methods, however, may be implemented to limit three dimensional ambiguities to the two dimensional case.

In one method, a planar array of antennas may be utilized instead of a simple linear array. A second method would be to rotate the linear array

in a plane not including the source and preferably perpendicular to the source. In both cases cylindrical symmetry would be eliminated and only the two dimensional ambiguities described in the following section would exist.

It could be said that the National Radio Astronomy Observatory "Very Large Array" aperture synthesis radio telescope utilized both techniques. Its planar "Y" configuration was effectively rotated with respect to astronomical sources by the Earth's motion due to the latitude of the VLA site in New Mexico. If the VLA site had been located at the north or south poles, only a simple linear array would have been required. However, at the lower latitude in New Mexico, only limited rotation was achieved for non-polar sources and thus three linear arrays in a "Y" shaped configuration provided near complete baseline coverage of radio sources in a range of directions.

For satellite based observations of the earth from a geostationary satellite, the rotational method would probably be the most economical, since a single linear array could be used to synthesize a planar array, and generate all baselines required to meet resolution requirements, ~~In fact, a near infinite number of baselines could be achieved~~ at no additional cost in microwave hardware ~~and would~~ also thus minimiz~~e~~ing satellite payload.

2. Two Dimensional Ambiguities

In this case, ambiguities are restricted to the plane containing the two antennas and the source. Thus ϕ_s is a constant defining that plane.

Therefore, the coordinates of the desired source are given by $(r=\infty, \theta=\theta_s, \phi=\phi_s)$, where ϕ_s is considered a constant. The delay function τ is again given by:

$$\tau = \frac{s}{c}\cos\theta_s$$

where τ = delay of time of arrival of signal at antenna B relative to antenna A.

s = spacing between the two antennas in meters.

c = speed of light in meters/sec

θ_s = angle to source measured from the positive z axis.

From the trigonometric identity

$$\cos(\alpha) = \cos(-\alpha)$$

then, $\tau = \frac{s}{c}\cos(-\theta_s)$

Therefore $(r=\infty, \theta=-\theta_s, \phi=\phi_s)$ is ambiguous with $(r=\infty, \theta=\theta_s, \phi=\phi_s)$ since the same delay results from the two directions.

Note that the three dimensional case includes the two dimensional case. In three dimensions,

$(r=\infty, \theta=\theta_s, \phi\neq\phi_s)$ is ambiguous with $(r=\infty, \theta=\theta_s, \phi=\phi_s)$

therefore

$(r=\infty, \theta=\theta_s, \phi=\phi_s+\pi)$ is ambiguous with $(r=\infty, \theta=\theta_s, \phi=\phi_s)$

but

$(r=\infty, \theta=\theta_s, \phi=\phi_s+\pi)$ is equivalent to $(r=\infty, \theta=-\theta_s, \phi=\phi_s)$.

Therefore $(r=\infty, \theta=-\theta_s, \phi=\phi_s)$ is ambiguous with $(r=\infty, \theta=\theta_s, \phi=\phi_s)$, the previously derived two dimensional case.

Also, note that the delay $\tau = \frac{s}{c}\cos\theta$ is equivalent to $\tau = \frac{s}{c}\sin\psi$ presented in chapter II.

$$\tau = \frac{s}{c}\cos\theta$$

$$\tau = \frac{s}{c}\cos(\frac{\pi}{2}-\psi)$$

Using the trigonometric identity

$$\cos(\frac{\pi}{2}-\alpha) = \sin\alpha,$$

therefore, $\tau = \frac{s}{c}\sin\psi$. The ambiguities derived here in terms of θ, can similarly be converted to terms of ψ, if desired.

B. CORRELATOR AMBIGUITIES

In this section the complex correlator presented in Chapter II is shown to not measure time delay uniquely for a continuous source at any angle of arrival. However the maximum antenna spacing s_λ in wavelengths can be calculated for unambiguous operation of two element simple or delay tracking interferometers. The problem of ambiguous operation for large spacing interferometers is then shown to be utilized to advantage in unambiguous operation of aperture synthesis interferometers.

The complex correlator visibility functions for a two element interferometer in terms of time delays were previously derived in Chapter II as:

Simple Two-Element Interferometer

$$V_{complex} = \tfrac{1}{2}\cos\omega\tau(t) + j\tfrac{1}{2}\sin\omega\tau(t)$$

where $V_{complex}$ = complex visibility function

ω = received frequency in radians/seconds

$\tau(t)$ = time delay of arrival at the second antenna relative to the first antenna in seconds

Delay Tracking Two-Element Interferometer

$$V_{complex} = \tfrac{1}{2}\cos\omega(\tau_{ref}(t)-\tau(t)) + j\tfrac{1}{2}\sin\omega(\tau_{ref}(t)-\tau(t))$$

where $V_{complex}$ = complex visibility function

ω = received frequency in radians/seconds

$\tau(t)$ = time delay of arrival at the second antenna relative to the first antenna in seconds

Note that in both cases, the cosine and sine functions are orthogonal functions which are capable of uniquely defining the arguments only over a

π radian range individually or a 2π range when used together. Thus if $|\omega\tau(t)| < \pi$ then the complex correlator will provide an unambiguous visibility function for the simple interferometer.

Delay tracking can be seen to provide an advantage in this limitation if the peak field of view is small, since $\tau_{ref}(t) \approx \tau(t)$.

If $|\omega(\tau_{ref}(t) - \tau(t))| < \pi$,

then the complex correlator will provide an unambiguous visibility function for the delay tracking interferometer. If the peak field of view of the object is small then $\tau_{ref}(t) \approx \tau(t)$ and a much larger $\tau(t)$ can be accommodated before an ambiguity will result.

The effect of antenna to antenna spacing can most easily be examined in general by writing the complex visibility function as a function of spacing and spatial frequency. In this case,

$$V_{complex} = \tfrac{1}{2}\cos(2\pi f_{s\lambda} s_\lambda) + j\tfrac{1}{2}\sin(2\pi f_{s\lambda} s_\lambda)$$

where $V_{complex}$ = complex visibility function

$f_{s\lambda}$ = spatial frequency, dimensionless

s_λ = spacing between the two antennas in number of wavelengths, dimensionless

2π = constant with dimensions of radians

The cosine and sine functions are orthogonal functions which are capable of uniquely defining the argument, $2\pi f_{s\lambda} s_\lambda$, only over a 2π radian range when used together. Therefore $|2\pi f_{s\lambda} s_\lambda| < \pi$ will result in unambiguous operation. Therefore, the range of antenna spacings allowed for an unambiguous visibility function is given by

$$s_\lambda < \frac{1}{2|f_{s\lambda}|}$$

By utilizing the table of maximum spatial frequencies derived in Figure III-1, along with the formula for unambiguous spacings, a table of

maximum spacings for each case of two-element interferometer and possible field of view can be derived. This table of unambiguous spacings is presented in Figure IV-2.

The delay tracking interferometer is seen to have an advantage over the simple interferometer in the restricted rotating case for objects with a small field of view, $\Delta\psi_{fv}$. The maximum spacing in this case may be increased by $\frac{1}{2|\sin(\frac{1}{2}\Delta\psi_{fv})|}$ from that of the simple interferometer and yet still maintain unambiguous operation.

However, the maximum spacing of an aperture synthesis interferometer must be set by the desired resolution angle, ψ_{resmin}, and not the much larger peak field of view angle, $\Delta\psi_{fv}$. As described in section III-B, the maximum spacing required for an aperture synthesis interferometer is:

$$s_{\lambda max} = \frac{1}{\psi_{res}}$$

but $\psi_{res} << \psi_{fv}$, for a high resolution instrument. Therefore,

$$s_{\lambda max} = \frac{1}{\psi_{resmin}} >> \frac{1}{2|fs_{\lambda}|}$$

for any of the cases shown in Figure IV-2. This will certainly be ambiguous in a two element fixed spacing interferometer, and will result in many cycles of the sinusoidal visibility function as shown in Figure IV-3.

While the correlator will produce an ambiguous visibility function output for a two-element interferometer, this problem is utilized to advantage in an aperture synthesis interferometer. By varying the spacing, through either physically moving one antenna relative to the other, rotating the two antennas in the plane of the source to change the projection of the baseline perpendicular to the directory of the source, utilizing discrete elemental antennas at strategic spacings to evenly sample the visibility function, or a combination of the above, the visibility function

UNAMBIGUOUS ANTENNA SPACING
FOR 2-ELEMENT INTERFEROMETER

SIMPLE $f_{s\lambda} = \sin(\omega_{ref} \cdot t + \Delta\psi)$

CASE	$\omega_{ref} \cdot t$	$\Delta\psi_{max}$	$s_\lambda < \frac{1}{2\lvert f_{s\lambda}\rvert}$
General Rotating	Any	Any	$< \frac{1}{2}$
Restricted Rotating	Any	$\Delta\psi_{fv}$	$< \frac{1}{2}$
General Stationary	0	Any	$< \frac{1}{2}$
Restricted Stationary	0	$\Delta\psi_{fv}$	$< \frac{1}{2\lvert \sin(\Delta\psi_{fv})\rvert}$

DELAY TRACKING $f_{s\lambda} = 2\cos(\omega_{ref} \cdot t + \frac{1}{2}\Delta\psi)\sin(\frac{1}{2}\Delta\psi)$

CASE	$\omega_{ref} \cdot t$	$\Delta\psi_{max}$	$s_\lambda < \frac{1}{2\lvert f_{s\lambda}\rvert}$
General Rotating	Any	Any	$< \frac{1}{4}$
Restricted Rotating	Any	$\Delta\psi_{fv}$	$< \frac{1}{4\lvert \sin(\frac{1}{2}\Delta\psi_{fv})\rvert}$
General Stationary	0	Any	$< \frac{1}{2}$
Restricted Stationary	0	$\Delta\psi_{fv}$	$< \frac{1}{2\lvert \sin(\Delta\psi_{fv})\rvert}$

FIGURE IV-2

TWO-ELEMENT INTERFEROMETER UNAMBIGUOUS SPACINGS DUE TO CORRELATOR

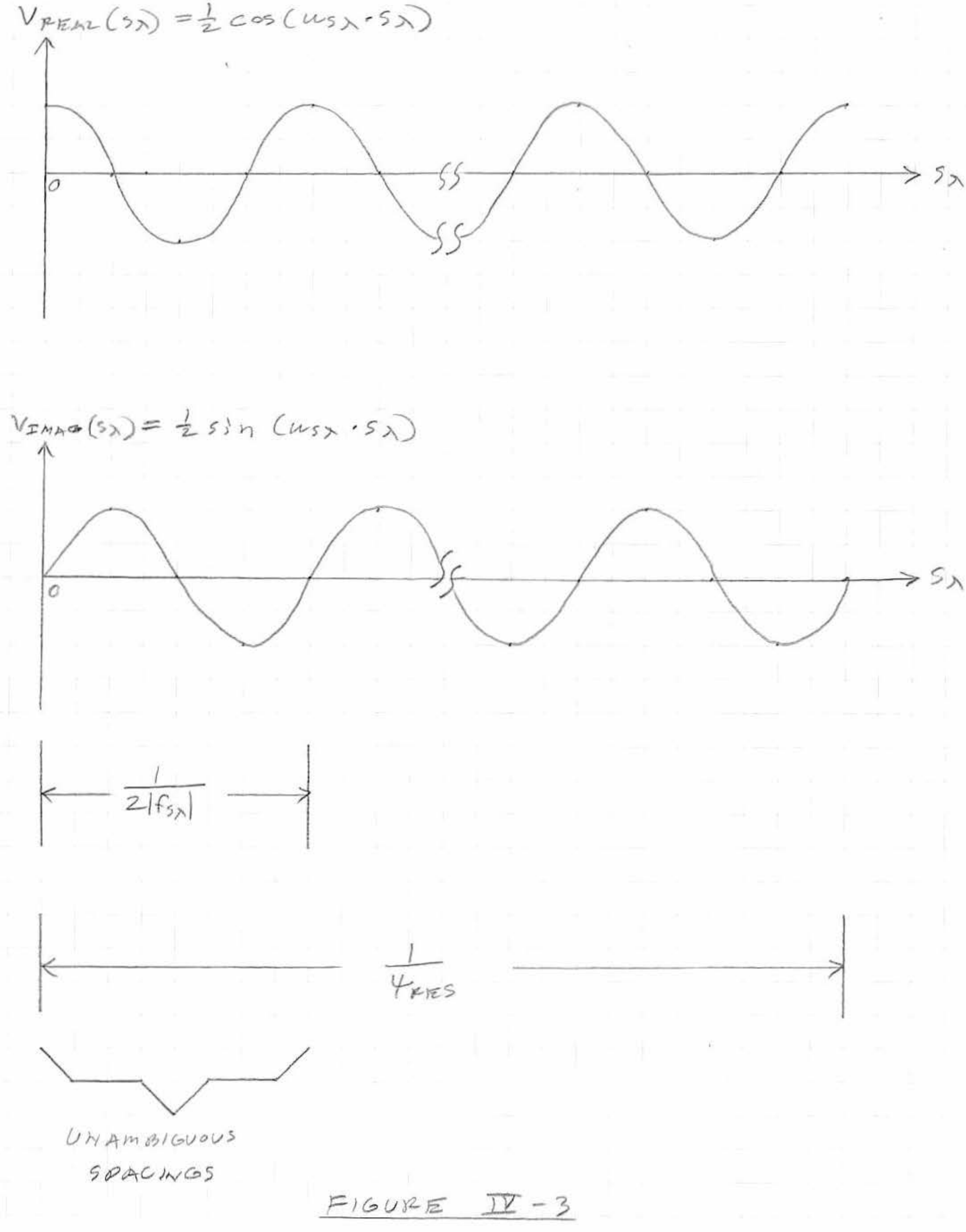


FIGURE IV-3

will appear as a periodic sinusoidal function versus spacing. By utilizing a Fourier processor complex objects may be mapped without correlator ambiguities in an aperture synthesis interferometer.

C. UNDERSAMPLING FOR A GIVEN FIELD OF VIEW

As discussed in section III-C, the Nyquist sampling theorem requires a band limited periodic spatial signal unbounded in the spatial domain. Both of these conditions do not hold for the reasons previously presented. Thus an error analysis is required for a given system to determine the actual number of samples and elemental antenna pattern required to limit the worst case error due to sidelobes resulting from aliasing.

D. FOURIER TRANSFORM WINDOWING AND MAPPING TECHNIQUES

The inverse Fourier transform required for processing the spatial domain is limited by the spacings $s_{\lambda min}$ to $s_{\lambda max}$. Techniques are available called "windowing" which minimize sidelobe effects due to this limitation on the inverse Fourier transform and must be investigated, again with a suitable error analysis. It may be possible to adapt mapping techniques such as "CLEAN" from radio astronomy to aid in this process.

V. ORBIT PARAMETERS

A. VIEW ANGLE RESOLUTION

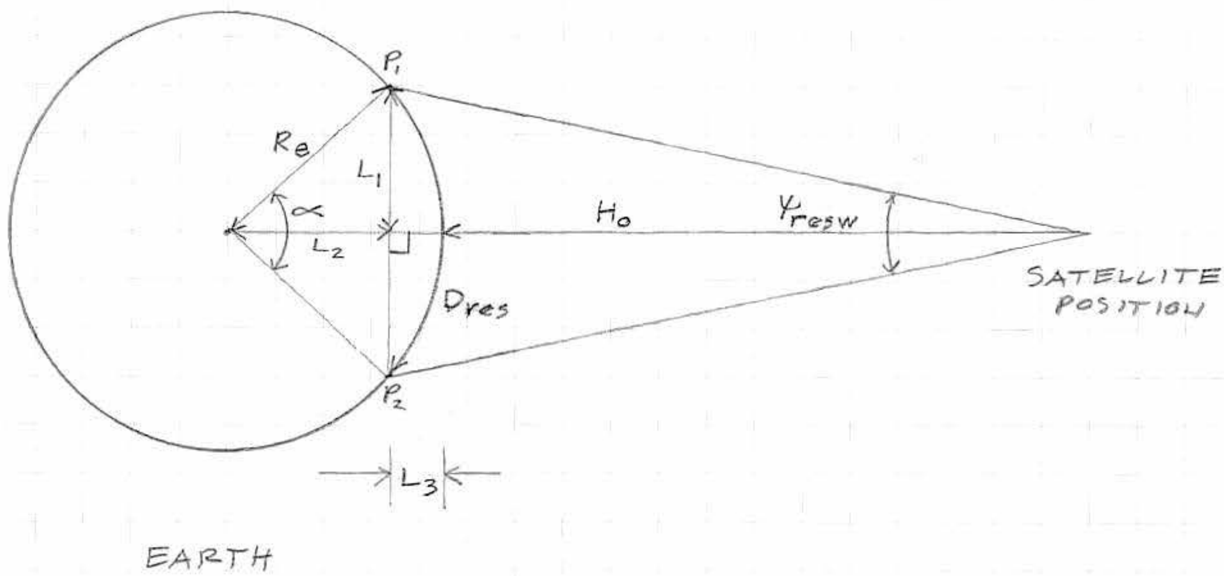
THE VIEW ANGLE RESOLUTION IS DEFINED AS THE ANGULAR RESOLUTION IN RADIANS REQUIRED FOR A GIVEN ARC LENGTH RESOLUTION DISTANCE ON THE EARTHS SURFACE DIRECTLY BELOW THE SATELLITE INTERFEROMETER. REFER TO FIGURE V - 1.

$$\alpha = \frac{D_{res}}{R_e}$$

WHERE α = ANGLE IN RADIANS AS DEFINED IN FIGURE V - 1

D_{res} = REQUIRED ARC LENGTH RESOLUTION DISTANCE IN KILOMETERS AT EARTH'S SURFACE

R_e = RADIUS OF EARTH EXPRESSED IN KILOMETERS



WELL ANGLE RESOLUTION

FIGURE V-1

$$L_1 = R_e \sin\left(\tfrac{1}{2}\alpha\right)$$

WHERE L_1 = DISTANCE IN KILOMETERS AS DEFINED IN FIGURE V-1

R_e = RADIUS OF EARTH IN KILOMETERS

α = ANGLE IN RADIANS AS DEFINED IN FIGURE V-1

$$L_2 = R_e \cos\left(\tfrac{1}{2}\alpha\right)$$

WHERE L_2 = DISTANCE IN KILOMETERS AS DEFINED IN FIGURE V-1

R_e = RADIUS OF EARTH IN KILOMETERS

α = ANGLE IN RADIANS AS DEFINED IN FIGURE V-1

$$L_3 = R_e - L_2$$

WHERE L_2, L_3 = DISTANCES IN KILOMETERS AS DEFINED IN FIGURE V-1

R_e = RADIUS OF EARTH IN KILOMETERS

$$\frac{1}{2}\psi_{resw} = \arctan\left(\frac{L_1}{H_0 + L_3}\right)$$

WHERE ψ_{resw} = WELL ANGLE RESOLUTION IN RADIANS

L_1, L_3 = DISTANCES IN KILOMETERS AS DEFINED BY FIGURE V-1

H_0 = HEIGHT IN KILOMETERS OF SATELLITE ORBIT ABOVE EARTH'S SURFACE

SUBSTITUTING FOR L_3,

$$\psi_{resw} = 2 \arctan\left(\frac{L_1}{H_0 + R_e - L_2}\right)$$

WHERE ψ_{resw} = WELL ANGLE RESOLUTION IN RADIANS

L_1, L_2 = DISTANCES IN KILOMETERS AS DEFINED BY FIGURE V-1

H_0 = HEIGHT IN KILOMETERS OF SATELLITE ORBIT ABOVE EARTH'S SURFACE

R_e = RADIUS OF EARTH IN KILOMETERS

SUBSTITUTING FOR L_1 AND L_2,

$$\psi_{resw} = 2 \arctan\left(\frac{R_e \sin(\frac{1}{2}\alpha)}{H_o + R_e - R_e \cos(\frac{1}{2}\alpha)} \right)$$

WHERE ψ_{resw} = VIEW ANGLE RESOLUTION IN RADIANS

R_e = RADIUS OF EARTH IN KILOMETERS

α = ANGLE IN RADIANS AS DEFINED BY FIGURE V - 1

H_o = HEIGHT IN KILOMETERS OF SATELLITE ORBIT ABOVE EARTH'S SURFACE

SUBSTITUTING FOR α,

$$\psi_{resw} = 2 \arctan\left(\frac{R_e \sin\left(\frac{D_{res}}{2 R_e}\right)}{H_o + R_e \left[1 - \cos\left(\frac{D_{res}}{2 R_e}\right)\right]} \right)$$

WHERE ψ_{resw} = VIEW ANGLE RESOLUTION IN RADIANS

R_e = RADIUS OF EARTH IN KILOMETERS

D_{res} = REQUIRED ARC LENGTH RESOLUTION DISTANCE IN KILOMETERS AT EARTH'S SURFACE.

H_0 = HEIGHT IN KILOMETERS OF SATELLITE ORBIT ABOVE EARTH'S SURFACE

B. BEST CASE ANGULAR FIELD OF VIEW.

THE ANGULAR FIELD OF VIEW MUST BE LIMITED BY EACH ELEMENTAL ANTENNA TO LIMIT THE MAXIMUM SPATIAL FREQUENCY AND THUS PREVENT UNDERSAMPLING IN THE SPATIAL DOMAIN AS DESCRIBED EARLIER. IF THREE ASSUMPTIONS ARE MADE IT IS POSSIBLE TO CALCULATE A BEST CASE FIELD OF VIEW BASED ON AN ASSUMED MAXIMUM HEIGHT OF THE ATMOSPHERE.

FIRST ASSUME A COLD SKY BACKGROUND WITH NO INTERFERING COSMIC RADIO SOURCES, ETC.

SECOND ASSUME THE ELEMENTAL ANTENNAS ARE PERFECT, THAT IS THEY HAVE A RECTANGULAR

RESPONSE VERSUS ANGLE AND THUS PERMIT ONLY RADIATION WITHIN A CERTAIN FIELD OF VIEW ANGLE TO PASS TO THE INTERFEROMETER UNIMPEDED. RADIATION OUTSIDE OF THIS FIELD OF VIEW IS ASSUMED TO BE INFINITELY ATTENUATED. IN REALITY THE ELEMENTAL ANTENNAS WILL PROBABLY HAVE A NEAR GAUSIAN RESPONSE WITH POSSIBLE SIDELOBES AND A DETAILED MEASUREMENT ERROR ANALYSIS WOULD BE REQUIRED TO FIND WHAT FIELD OF VIEW ANGLE COULD BE TOLERATED FOR A GIVEN ELEMENTAL ANTENNA PATTERN AND REQUIRED MAXIMUM ALIASING ERROR.

THIRD ASSUME THE ATMOSPHERE DOES NOT CONTRIBUTE ANY SIGNIFICANT TEMPERATURE ABOVE A SPECIFIC ALTITUDE, H_a IN KILOMETERS. WHILE IN REALITY THE ATMOSPHERIC PRESSURE, AND THUS TEMPERATURE, FALLS OFF GRADUALLY, A VALUE OF H_a COULD STILL BE ASSUMED SUCH THAT A GIVEN REQUIRED MEASUREMENT ERROR DUE TO ALIASING COULD BE MET.

BECAUSE OF THE ABOVE LIMITATIONS TO THE ASSUMPTIONS GIVEN, THE FOLLOWING PREDICTION OF FIELD OF VIEW ANGLE IS ASSUMED TO BE BEST CASE.

REFER TO FIGURE V-2.

SINCE THE FIELD OF VIEW IS LIMITED BY THE ASSUMED MAXIMUM HEIGHT OF THE ATMOSPHERE, THEN

$$\Delta \psi_{fv} = \arcsin\left(\frac{R_e + H_a}{R_e + H_o}\right)$$

WHERE $\Delta \psi_{fv}$ = ONE HALF OF THE TOTAL FIELD OF VIEW ANGLE IN RADIANS

R_e = RADIUS OF EARTH IN KILOMETERS

H_a = HEIGHT OF ATMOSPHERE IN KILOMETERS.

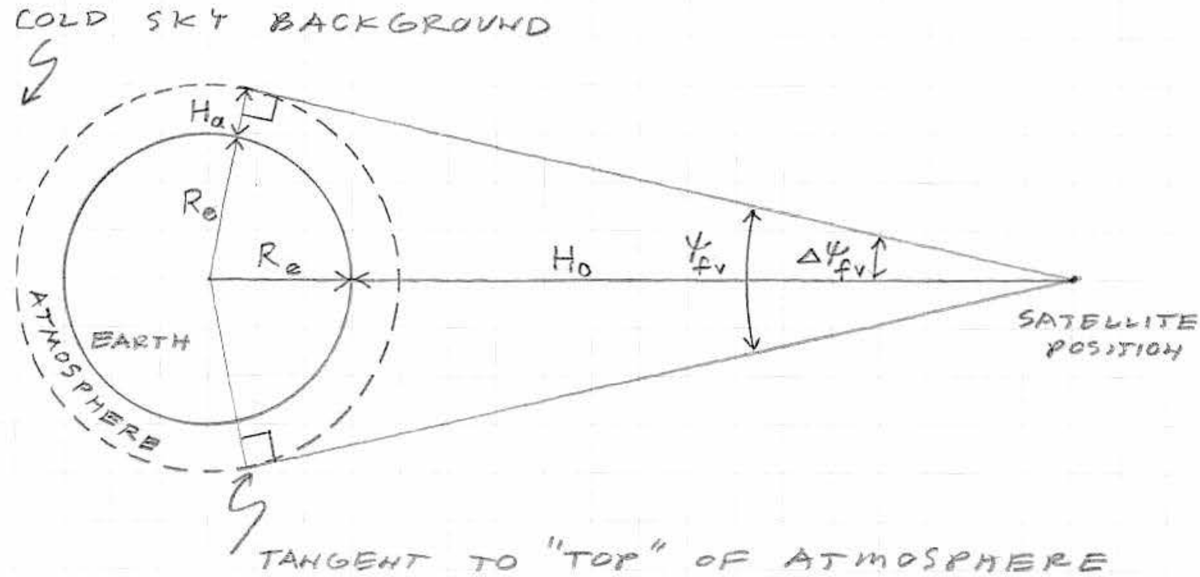
H_o = HEIGHT OF SATELLITE'S ORBIT IN KILOMETERS

NOTE THAT

$$\psi_{fv} = 2 \Delta \psi_{fv}$$

WHERE ψ_{fv} = TOTAL FIELD OF VIEW ANGLE IN RADIANS

$\Delta \psi_{fv}$ = ONE HALF OF THE TOTAL FIELD OF VIEW ANGLE IN RADIANS



BEST CASE ANGULAR FIELD OF VIEW

FIGURE V - 2

THEREFORE,

$$\psi_{fv} = 2 \arcsin\left(\frac{R_e + H_a}{R_e + H_o}\right)$$

WHERE ψ_{fv} = FIELD OF VIEW ANGLE IN RADIANS

R_e = RADIUS OF EARTH IN KILOMETERS

H_a = HEIGHT OF ATMOSPHERE IN KILOMETERS

H_o = HEIGHT OF SATELLITE'S ORBIT IN KILOMETERS.

AGAIN NOTE THAT THIS IS A BEST CASE ANALYSIS.

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ELEMENTAL ANTENNA DISCUSSION CONT'D

HOWEVER, THE ~~APPROXIMATE~~ BEST CASE PHYSICAL DIMENSIONS AND GAIN OF EACH ELEMENTAL ANTENNA CAN BE DETERMINED IN THE FOLLOWING ANALYSIS. AGAIN A BEST CASE ANALYSIS WILL BE ASSUMED TO CONFORM WITH THE EARLIER ARRAY PREDICTIONS.

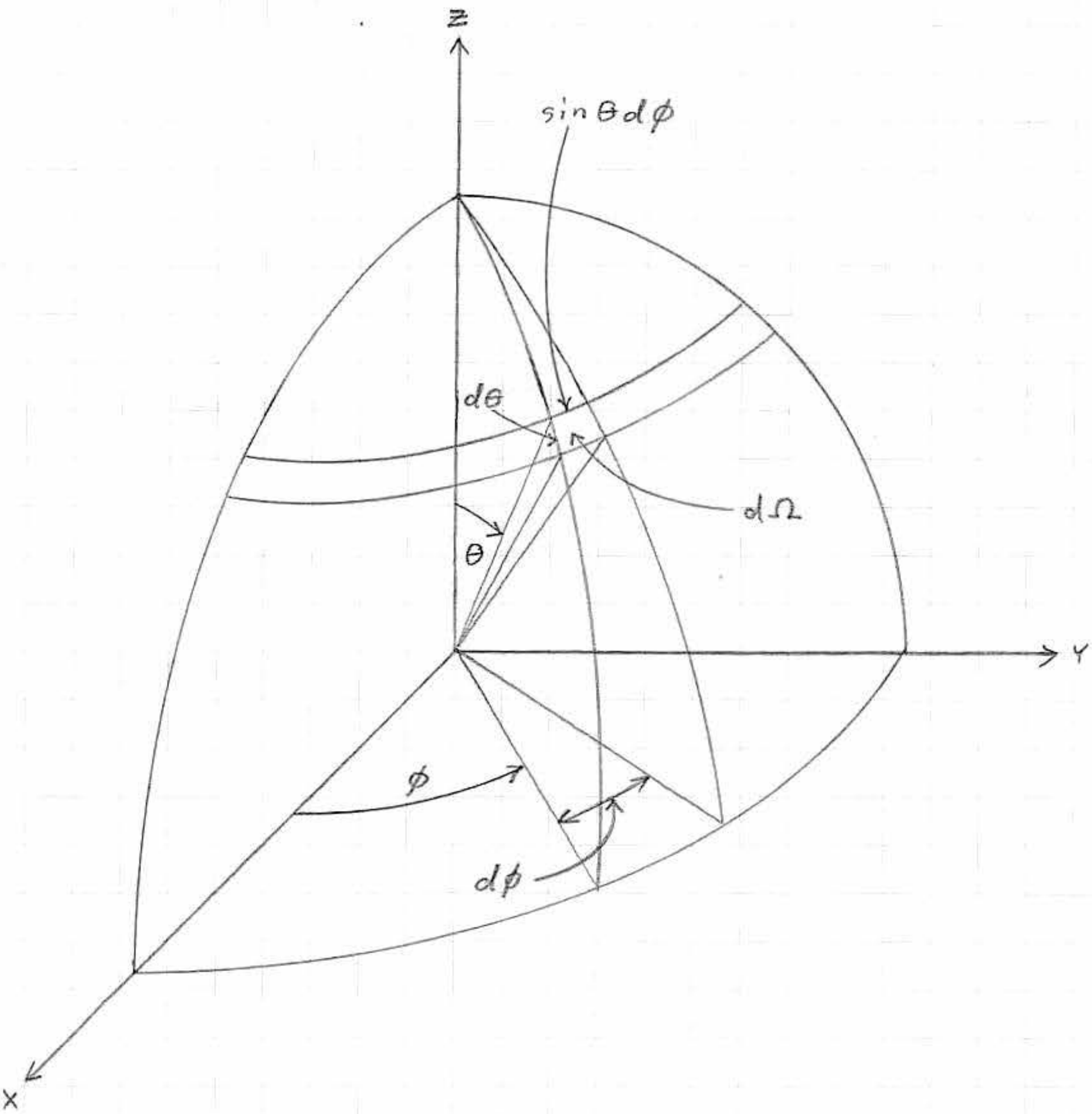
FIRST THE SOLID ANGLE OF AN ISOTROPIC ANTENNA IS DETERMINED USING FIGURE III-7. AN ISOTROPIC ANTENNA RECEIVES RADIATION EQUALLY FROM ALL DIRECTIONS. THE INFINITESIMAL SOLID ANGLE CORRESPONDING TO AN ISOTROPIC ANTENNA IS GIVEN BY

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

WHERE $d\Omega$ = INFINITESIMAL SOLID ANGLE IN RADIANS^2 OR STERADIANS

θ, ϕ = SPHERICAL COORDINATE ANGLES IN RADIANS AS DEFINED IN FIGURE III-7

PLANETARY



GEOMETRY FOR CALCULATION OF ISOTROPIC ANTENNA TOTAL SOLID ANGLE

FIGURE III - 7

THEREFORE THE TOTAL SOLID ANGLE OF AN ISOTROPIC ANTENNA IS GIVEN BY,

$$\Omega_I = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \sin\theta \, d\theta \, d\phi$$

THE INTEGRAL WITH RESPECT TO θ CAN BE REDUCED AS FOLLOWS;

$$\int_{\theta=0}^{\theta=\pi} \sin\theta \, d\theta = -\cos\theta \Big|_0^{\pi} = 2$$

THEREFORE,

$$\Omega_I = \int_{\phi=0}^{\phi=2\pi} 2 \, d\phi = 2\phi \Big|_0^{2\pi} = 4\pi$$

THEREFORE THE TOTAL SOLID ANGLE OF AN ISOTROPIC ANTENNA IS GIVEN BY 4π RADIANS^2 OR 4π STERADIANS.

THE DIRECTIVE GAIN OF ANY ANTENNA IS GIVEN BY THE THE SOLID ANGLE OF A SPHERE, 4π RADIANS², DIVIDED BY THE ANTENNA TOTAL PATTERN SOLID ANGLE AS SHOWN IN FIGURE III - 8. (KRAUSS P. 156)

THEREFORE:

$$D = \frac{4\pi}{\Omega_A}$$

WHERE = DIRECTIVE GAIN, DIMENSIONLESS

4π = SOLID ANGLE OF A SPHERE IN RADIANS²

Ω_A = TOTAL PATTERN SOLID ANGLE OF THE ANTENNA IN RADIANS²

THE PATTERN SOLID ANGLE IS THE SUM OF THE MINOR LOBE SOLID ANGLE AND THE MAIN BEAM SOLID ANGLE. (KRAUSS P. 154)

FIGURE III -8

$$\Omega_A = \Omega_M + \Omega_m$$

WHERE Ω_A = TOTAL PATTERN SOLID ANGLE IN RADIANS^2

Ω_M = MAIN BEAM SOLID ANGLE IN RADIANS^2

Ω_m = MINOR LOBE SOLID ANGLE IN RADIANS^2

HOWEVER SIDE LOBES ARE UNDESIRABLE, BECAUSE OF POSSIBLE ADVERSE EFFECTS FROM SOURCES OUTSIDE THE FIELD OF VIEW OF THE INSTRUMENT,

[illegible] ASSUME THE ANTENNA HAS NO MINOR SIDE LOBES (IN OTHER WORDS THE ANTENNA HAS A GAUSSIAN POWER PATTERN), THEN

$$\Omega_A = \Omega_M$$

$$D = \frac{4\pi}{\Omega_M}$$

WHERE D = DIRECTIVE GAIN, DIMENSIONLESS

4π = SOLID ANGLE OF A SPHERE IN RADIANS^2

Ω_M = MAIN BEAM SOLID ANGLE IN RADIANS^2

THE MAIN BEAM SOLID ANGLE CAN BE EXPRESSED IN TERMS OF HALF POWER BEAMWIDTHS. (KRAUSS - P. 221)

$$\Omega_M = K_p \, \theta_{HP} \, \phi_{HP}$$

WHERE Ω_M = MAIN BEAM SOLID ANGLE IN RADIANS^2

K_p = FACTOR BETWEEN 1.0 FOR A UNIFORM APERTURE DISTRIBUTION TO 1.13 FOR A GAUSSIAN POWER PATTERN.

θ_{HP} = HALF POWER BEAM WIDTH IN θ PLANE IN RADIANS

ϕ_{HP} = HALF POWER BEAM WIDTH IN ϕ PLANE IN RADIANS.

NO

ASSUME A GAUSSIAN POWER PATTERN ~~AN UNIFORM APERTURE DISTRIBUTION~~, IN OTHER WORDS

$K_p = \cancel{1.0}$ 1.13 FOR NO SIDE LOBES.

THEN

$$\Omega_M = k_p \theta_{HP} \phi_{HP}$$

AND

$$D = \frac{4\pi}{k_p \theta_{HP} \phi_{HP}}$$

THE ISOTROPIC GAIN IS DEFINED AS

$$G = D K_o$$

WHERE G = POWER GAIN OVER AN ISOTROPIC ANTENNA, DIMENSIONLESS

D = DIRECTIVE GAIN, DIMENSIONLESS

K_o = OHMIC LOSS FACTOR, DIMENSIONLESS ($0 \leq K_o \leq 1$)

ASSUME THE OHMIC LOSS FACTOR TO BE UNITY.

THEN

$$G = D$$

AND

$$G = \frac{4\pi}{K_p \theta_{HP} \phi_{HP}}$$

IF A CIRCULAR APERTURE IS ASSUMED WITH THE HALF POWER BEAMWIDTH EQUAL TO THE FIELD OF VIEW ANGLE.

$$\Psi_{FV} = \phi_{HP} = \theta_{HP}$$

AND

$$G = \frac{4\pi}{K_p (\Psi_{FV})^2}$$

WHERE G = POWER GAIN OF AN ELEMENTAL ANTENNA OVER AN ISOTROPIC ANTENNA, DIMENSIONLESS

4π = CONSTANT WITH UNITS RADIANS^2

Ψ_{FV} = FIELD OF VIEW ANGLE IN RADIANS

FROM KRAUSS, p.157

$$\lambda^2 = A_e \Omega_A$$

WHERE λ = WAVELENGTH IN METERS

A_e = EFFECTIVE APERTURE IN METERS SQUARED

Ω_A = BEAM SOLID ANGLE RADIANS SQUARED

THE APERTURE EFFICIENCY IS DEFINED AS, KRAUSS p.213

$$\epsilon_{ap} = \frac{A_e}{A_p}$$

WHERE ϵ_{ap} = APERTURE EFFICIENCY

A_e = EFFECTIVE APERTURE

A_p = PHYSICAL APERTURE

THEREFORE

$$\lambda^2 = \epsilon_{ap} A_p \Omega_A$$

ASSUMING NO MINOR LOBES

$\Omega_M = \Omega_A$

WHERE Ω_M = SOLID ANGLE OF MAIN BEAM

Ω_A = TOTAL SOLID ANGLE OF ANTENNA

THEN

$$\lambda^2 = \epsilon_{Ap} A_p \Omega_M$$

ASSUMING $\Omega_M = k_p 4_{FV}^2$ AS BEFORE,

$$\lambda^2 = \epsilon_{Ap} A_p 4_{FV}^2 k_p$$

THEN

$$A_p = \frac{\lambda^2}{k_p \epsilon_{Ap} 4_{FV}^2}$$

WHERE A_p = PHYSICAL APERTURE OF ANTENNA IN METERS SQUARED

λ = WAVELENGTH IN METERS

ϵ_{Ap} = APERTURE EFFICIENCY DIMENSIONLESS

ASSUMING A CIRCULAR BEAMWIDTH

FROM A CIRCULAR APERTURE

$$A_P = \pi\left(\tfrac{1}{2}d\right)^2 = \frac{\pi d^2}{4}$$

WHERE A_P = PHYSICAL APERTURE OF ANTENNA IN METERS SQUARED

d = DIAMETER OF PHYSICAL APERTURE IN METERS

$$\frac{\pi d^2}{4} = \frac{\lambda^2}{\epsilon_{AP}(\psi_{FV})^2 K_P}$$

$$d = \sqrt{\frac{4\lambda^2}{\pi\,\epsilon_{AP}(\psi_{FV})^2 K_P}} = \frac{2\lambda}{K_P\,\psi_{FV}\sqrt{\pi\,\epsilon_{AP}}}$$

WHERE d = DIAMETER OF ELEMENTAL ANTENNA PHYSICAL APERTURE IN METERS

λ = WAVELENGTH IN METERS

ϵ_{AP} = APERTURE EFFICIENCY, DIMENSIONLESS

ψ_{FV} = FIELD OF VIEW ANGLE IN RADIANS.

E. COMPARISON WITH SINGLE LARGE PARABOLIC DISH ANTENNA

BY SUBSTITUTING ψ_{RES} WITH ψ_{FV} THE DIAMETER OF A SINGLE LARGE DISH ANTENNA CAN BE CALCULATED IN A SIMILAR MANNER AS THAT FOR THE ELEMENTAL ANTENNA.

IN THIS CASE

$$G = \frac{4\pi}{k_p (\psi_{RES})^2}$$

WHERE G = ISOTROPIC POWER GAIN OF A SINGLE DISH ANTENNA TO ACHIEVE RESOLUTION, DIMENSIONLESS.

4π = CONSTANT WITH UNITS OF RADIANS SQUARED.

ψ_{RES} = RESOLUTION ANGLE IN RADIANS

SIMILARLY,

$$A_p = \frac{\lambda^2}{k_p \epsilon_{AP} (\Psi_{RES})^2}$$

WHERE A_p = PHYSICAL APERTURE OF SINGLE LARGE DISH ANTENNA TO ACHIEVE RESOLUTION IN METERS SQUARED

λ = WAVELENGTH IN METERS

ϵ_{AP} = APERTURE EFFICIENCY DIMENSIONLESS

Ψ_{RES} = REQUIRED RESOLUTION ANGLE IN RADIANS.

ALSO

$$d = \frac{2\lambda}{k_p \Psi_{RES} \sqrt{\pi \epsilon_{AP}}}$$

WHERE d = DIAMETER OF SINGLE LARGE DISH ANTENNA TO ACHIEVE RESOLUTION IN METERS SQUARED.

λ = WAVELENGTH IN METERS

Ψ_{RES} = REQUIRED RESOLUTION ANGLE IN RADIANS

ϵ_{AP} = APERTURE EFFICIENCY, DIMENSIONLESS

CWD

NOTE FOR A GAUSSIAN POWER PATTERN $K_p = 1.13$

$$d = \frac{2\lambda}{1.13\sqrt{\pi}\,\epsilon_{AP}\,\psi_{RES}}$$

ASSUME A BEST CASE APERTURE EFFICIENCY $\epsilon_{AP} = 1.0$

$$d = \frac{2}{1.13\sqrt{\pi}}\,\frac{\lambda}{\psi_{RES}}$$

$$d = 1.00\,\frac{\lambda}{\psi_{RES}} = \frac{\lambda}{\psi_{RES}}$$

$$\therefore\ \psi_{RES} = \frac{\lambda}{d}$$

NOTE THAT THIS IS THE SAME NUMBER FOR ψ_{RES} AS IN THE NON REDUNDANT LINEAR ARRAY CASE

$$\psi_{RES} = \frac{\lambda}{S} = \frac{1}{S_\lambda}$$

10/14/86 - 1

FOURIER TRANSFORM

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j 2\pi f t}\, dt$$

INVERSE FOURIER TRANSFORM

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j 2\pi f t}\, df$$

IN SPATIAL COORDINATES,

LET $t \longrightarrow s_\lambda$

$f \longrightarrow f$ (GENERAL SPATIAL FREQUENCY)

$\therefore$ FOURIER TRANSFORM

$$X(f) = \int_{-\infty}^{\infty} x(s_\lambda)\, e^{-j 2\pi f \cdot s_\lambda}\, ds_\lambda$$

INVERSE FOURIER TRANSFORM

$$x(s_\lambda) = \int_{-\infty}^{\infty} X(f)\, e^{j 2\pi f \cdot s_\lambda}\, df$$

10/14/86 - 2

LET $\mathfrak{X}(f_{s\lambda}) = A^2\delta(f_{s\lambda})$

THEN $x(s\lambda) = \int_{-\infty}^{\infty} A^2\delta(f_{s\lambda}) e^{j2\pi f \cdot s\lambda}\, df$

$$x(s\lambda) = A^2 e^{j2\pi(0)\cdot s\lambda}$$

$$x(s\lambda) = A^2 \cdot 1 = A^2$$

SPECIFIC FREQ,

NOW LET $x(s\lambda) = A^2 e^{j2\pi f_{s\lambda}\cdot s\lambda}$

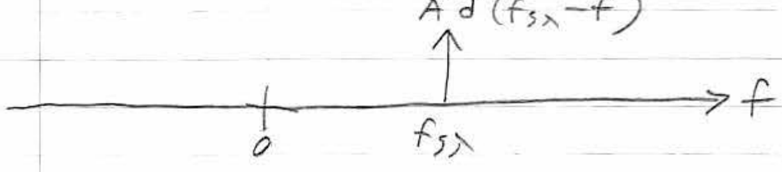
AMPLITUDE OF INCOMING SIGNAL

$$\mathfrak{X}(f) = \int_{-\infty}^{\infty} A^2 \left(e^{j2\pi f_{s\lambda}\cdot s\lambda}\right)\left(e^{-j2\pi f\cdot s\lambda}\right) ds\lambda$$

$$\mathfrak{X}(f) = \int_{-\infty}^{\infty} A^2\, e^{j2\pi(f_{s\lambda}-f)\cdot s\lambda}\, ds\lambda$$

BUT BY SUBSTITUTION

$$\mathfrak{X}(f) = A^2\delta(f_{s\lambda} - f)$$



10/14/86 - 3

NOW KNOW $f_{s\lambda}$!

SIMPLE INTERFEROMETER:

$$f_{s\lambda} = \sin(\omega_{REF} \cdot t + \Delta\varphi)$$

$$\therefore \boxed{\Delta\varphi = \sin^{-1}(f_{s\lambda}) - \omega_{REF} \cdot t}$$

DELAY TRACKING INTERFEROMETER:

$$f_{s\lambda} = 2\cos(\omega_{REF} \cdot t + \tfrac{1}{2}\Delta\varphi)\sin(\tfrac{1}{2}\Delta\varphi)$$

$$f_{s\lambda} = \sin(\omega_{REF} \cdot t + \Delta\varphi) - \sin(\omega_{REF} \cdot t)$$

$$f_{s\lambda} + \sin(\omega_{REF} \cdot t) = \sin(\omega_{REF} \cdot t + \Delta\varphi)$$

$$\boxed{\Delta\varphi = \sin^{-1}(f_{s\lambda} + \sin(\omega_{REF} \cdot t)) - \omega_{REF} \cdot t}$$